Lecture 6

Friday, April 10, 2020

Last time we discussed how to solve a general quadratic equation of the form $az^2+bz+c=0$. Now we will discuss how to solve a general cubic equation of the form $az^3+bz^2+cz+d=0$.

If this equation has a special root, say $z=z_0$, then one should factor out $z-z_0$, and then the problem becomes solving a quadratic equation (see an example given last time). Here we are interested in a situation where no special roots are known, for example, the equation

$$z^3+2z+2=0.$$

One can check that none of the numbers $\pm 1, \pm 2$ solve the equation.

We discuss in Lecture 2 how to solve the cubic equation of the form

$$z^3+pz+q=0. \qquad (*)$$

We refer to that method as Cardano's method (~ 1545). His idea is as follows: split $z=u+v$ where u and v are to be determined. The equation $(*)$ becomes

$$u^3+v^3+(3uv+p)(u+v)+q=0.$$

We then imposed an addition on u and v, namely $3uv+p=0$. Then we get a system of two unknowns:

$$\begin{cases} u^3+v^3+q=0 \\ 3uv+p=0. \end{cases}$$

From here we get

$$\begin{cases} u^3+v^3=-q, \\ u^3v^3=-p^3/27. \end{cases}$$

The $t=u^3$ and $s=v^3$ solve the quadratic equation

$$w^2+qw-p^3/27=0.$$

We get

$$w=\frac{1}{2}\left(-q\pm\sqrt{\Delta}\right) \qquad \text{where } \Delta=q^2+\frac{4p^3}{27}.$$

We get
$$u^3 = \frac{1}{2}\left(-q+\sqrt{\Delta}\right)$$
$$v^3 = \frac{1}{2}\left(-q-\sqrt{\Delta}\right)$$
Then
$$u = \sqrt[3]{\frac{1}{2}\left(-q+\sqrt{\Delta}\right)},$$
$$v = \sqrt[3]{\frac{1}{2}\left(-q-\sqrt{\Delta}\right)}.$$

Note that there are three values of u and three values of v because a complex number has three third roots. Let us denote the three values of u as u_1, u_2, u_3, and the three values of v as v_1, v_2, v_3. The roots $z = u+v$ of the original cubic equation are therefore

$$\begin{array}{lll} u_1+v_1, & u_2+v_1, & u_3+v_1, \\ u_1+v_2, & u_2+v_2, & u_3+v_2, \\ u_1+v_3, & u_2+v_3, & u_3+v_3. \end{array}$$

Among these 9 pairs of (u,v), we only take the ones such that $uv = -\frac{p}{3}$. One can check this relation by checking if $\arg(u)+\arg(v) = \arg(-\frac{p}{3})$ in modulo 2π. See the worksheet for an example.

If the equation is in a general form $az^3+bz^2+cz+d=0$, we can divide both sides by a:
$$z^3 + \frac{b}{a}z^2 + \frac{c}{a}z + \frac{d}{a} = 0$$
and then use the change of variable $x = z + \frac{b}{3a}$. The equation that x satisfies will be of the form $x^3+px+q=0$.

<u>Ex</u>:
$$z^3+3z^2+1=0$$
Change of variable $x = z+\frac{3}{3} = z+1$, or equivalently $z = x-1$.

Substitute this z into the equation:

$$(x-1)^3 + 3(x-1)^2 + 1 = 0$$

or equivalently,

$$x^3 - 3x + 3 = 0$$

We know how to solve this equation by Cardano's method.

[See Problem 2 on the worksheet for example.]

* <u>The exponential function with complex variable</u>

We know what e^x is when $x \in \mathbb{R}$. That is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

We know what e^{iy} is when $y \in \mathbb{R}$. That is

$$e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots$$

$$= \cos y + i \sin y.$$

How do we define e^z where $z = x + iy$?

There are two equivalent ways to define e^z. The first way is

$$e^z = e^{x+iy} \stackrel{\text{def}}{=} e^x e^{iy} = e^x \cos y + i e^x \sin y.$$

The second way is

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Why are these two ways equivalent to each other? Let us put $u = x$ and $v = iy$. We know that

$$e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$$

$$e^v = 1 + \frac{v}{1!} + \frac{v^2}{2!} + \frac{v^3}{3!} + \cdots$$

Then $$e^u e^u = \left(1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right)\left(1 + \frac{v}{1!} + \frac{v^2}{2!} + \frac{v^3}{3!} + \cdots\right)$$

$$= 1 + (u+v) + \left(\frac{u^2}{2} + uv + \frac{v^2}{2}\right) + \cdots$$

$$= 1 + \frac{u+v}{1!} + \frac{(u+v)^2}{2!} + \cdots$$

Thus, $\underbrace{e^x e^{iy}}_{\text{first def. of } e^z} = e^u e^v = 1 + \frac{u+v}{1!} + \frac{(u+v)^2}{2!} + \cdots = \underbrace{1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots}_{\text{second def. of } e^z}.$

How do we define sine and cosine of complex variable?

We know that $e^{ix} = \cos x + i \sin x$ for any $x \in \mathbb{R}$.

Now replace x by $-x$:

$$e^{-ix} = \cos(-x) + i\sin(-x)$$
$$= \cos x - i \sin x.$$

From here we get

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

These identities are true when x is a real number. We will now allow x to be any complex number. Thus,

$$\cos z \stackrel{\text{def}}{=} \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

These definitions are the same as

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

(One can check by writing the series for e^{iz} and e^{-iz}.)

See computation examples on the worksheet.