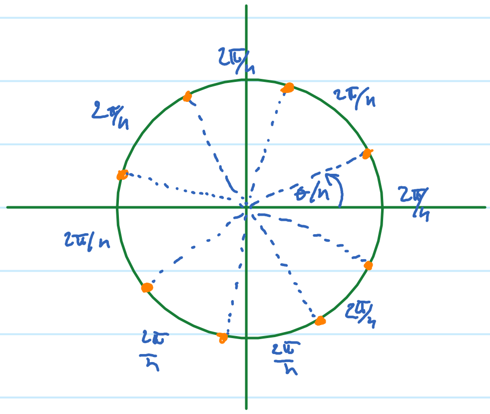
Lecture 5

Wednesday, April 8, 2020

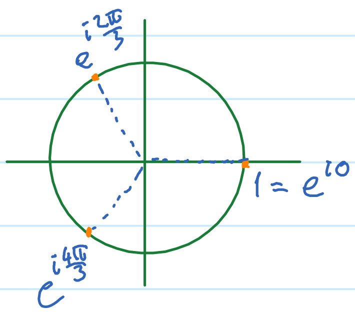
As we saw last time, any nonzero complex number has exactly n n'th roots. Zero has only one n'th root, which is zero. In polar form, $z = re^{i\theta}$ and

$$\sqrt[n]{z} = \underbrace{\sqrt[n]{r}}_{\text{usual } n\text{'th root of a real number}} e^{i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)} \qquad \text{for } k = 0, 1, 2, \dots, n-1.$$

Geometrically, these n'th roots of z lie on a circle of radius $\sqrt[n]{r}$ and are equally spaced, starting with angle $\frac{\theta}{n}$, forming a regular n-polygon.



Ex: Find all third roots of 1 and locate them on the complex plane.



In polar form, $1 = 1e^{i0}$. Thus,

$$\sqrt[3]{1} = \underbrace{\sqrt[3]{1}}_{\text{usual third root}} e^{i\left(\frac{0}{3} + k\frac{2\pi}{3}\right)} = e^{ik\frac{2\pi}{3}} \quad \text{for } k = 0, 1, 2.$$

We have seen that the arithmetic of complex numbers is very much the same as the arithmetic of real numbers. The number i was introduced somewhat artificially: a "number" satisfying $i^2 = -1$. In the following is an example of a structure that is essentially the same as complex numbers, where the imaginary unit is a natural (as oppose to artificial) element.

Consider the set of matrices

$$M = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

M doesn't contain all 2x2 matrices, only those of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

M has vector space structure:

- The sum of two elements of M is another element of M:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix} \in M.$$

- The (real) scaling of an element of M is another element of M:

$$\alpha \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \alpha a & -(\alpha b) \\ \alpha b & \alpha a \end{bmatrix} \in M.$$

M is also closed under multiplication:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix} \in M$$

M is also closed under division (inversion):

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} a/(a^2+b^2) & b/(a^2+b^2) \\ -b/(a^2+b^2) & a/(a^2+b^2) \end{bmatrix} \in M$$

We see that matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ behaves just like complex number $a+ib$. [One can check how similar M is to $\mathbb{C}$ by looking at the multiplication rule and inversion rule above.]

Matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ behaves like number $1 \in \mathbb{C}$ (the multiplication unit). Matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ behaves like number $i \in \mathbb{C}$ because

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Also,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{like "1"}} + b \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{like "}i\text{"}} \quad \text{is like "}a+bi\text{".}$$

Those who already took Math 343 can see that M is a field and is isomorphic to $\mathbb{C}$, i.e. having the same algebraic structure. In other words, M is another manifestation of complex numbers. The "imaginary" number i has become so "real".

* Solving quadratic equations:

Recall how we solved the quadratic equation

$$ax^2 + bx + c = 0$$

where $a, b, c \in \mathbb{R}$. The main idea is to complete the square:

$$\underbrace{x^2 + \frac{b}{a}x + \frac{c}{a}}_{= \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}} = 0$$

Then we get $\quad \left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2 - 4ac}{4a^2} = \dfrac{\Delta}{4a^2}. \qquad (*)$

This equation has two distinct real roots if $\Delta > 0$, a double root if $\Delta = 0$, and no real roots if $\Delta < 0$.

If we allow x to be complex, we don't need to worry about the sign of the discriminant Δ.

$$x + \frac{b}{2a} = \pm \frac{\sqrt{\Delta}}{2a}$$

or $\quad x = \dfrac{-b \pm \sqrt{\Delta}}{2a} \qquad$ (quadratic formula)

Here a, b, c can be complex numbers. Note that Δ has two complex square roots differing each other by a sign, one can write the above

formula as $$x = \frac{-b + \sqrt{\Delta}}{2a}$$
with the understanding that $\sqrt{\Delta}$ has two complex values.

Ex:

Solve for all complex roots of $z^2 + z + 4 = 0$.

We have $\Delta = 1^2 - 4 \times 4 = -15$.

$\sqrt{\Delta} = \sqrt{-15} = \pm i\sqrt{15}$.

Then $$z = \frac{-b + \sqrt{\Delta}}{2a} = \frac{-1 \pm i\sqrt{15}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{15}}{2}$$

Ex:

Solve for all complex roots of $z^3 - 2z^2 + 2z - 2 - i = 0$, with a hint that $z = i$ is one of the roots.

Because i is a root, $z - i$ is a factor of the given cubic polynomial. We will do long division:

$$\begin{array}{r l}
 & z^2 + (-2+i)z + (1-2i) \\
z - i \; \big) & z^3 - 2z^2 + 2z - 2 - i \\
- & \underline{(z^3 - iz^2)} \\
 & (-2+i)z^2 + 2z - 2 - i \\
- & \underline{[(-2+i)z^2 + (2i+1)z]} \\
 & (1-2i)z - 2 - i \\
- & \underline{[(1-2i)z - 2 - i]} \\
 & 0
\end{array}$$

Thus, $z^3 - 2z^2 + 2z - 2 - i = (z - i)(z^2 + (-2+i)z + 1 - 2i)$.

Of course $z = i$ is a root. To find other roots, we only need to solve the quadratic equation
$$z^2 + (-2+i)z + 1 - 2i = 0$$
We have
$$\Delta = (-2+i)^2 - 4(1-2i) = -1 + 4i.$$
To find the square roots of Δ, we write Δ in polar form:

$$\Delta = -1+4i = \underbrace{\sqrt{17}}_{r}\Big(\underbrace{-\frac{1}{\sqrt{17}}}_{\cos\theta} + i\underbrace{\frac{4}{\sqrt{17}}}_{\sin\theta}\Big) = re^{i\theta}$$

with $r = \sqrt{17}$ and $\theta = \arccos\left(-\frac{1}{\sqrt{17}}\right) \approx 1.8158$.

Then $\sqrt{\Delta} = \pm\sqrt[4]{17}\, e^{i\frac{\theta}{2}} = \pm\sqrt{17}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \approx \pm(1.2496 + i\,1.6005)$

One can easily continue from here.

Ex: Solve for all complex roots of the equation

$$z^4 + z^2 + 1 = 0.$$

Put $w = z^2$. Then we get $w^2 + w + 1 = 0$.

This equation has two roots: $w_1 = \dfrac{-1+i\sqrt{3}}{2}$ and $w_2 = \dfrac{-1-i\sqrt{3}}{2}$.

Taking the square roots of w_1 and w_2, we get z. There are four values of z, two coming from w_1 and two coming from w_2.

$$\sqrt{w_1} = \{z_1, z_2\},$$
$$\sqrt{w_2} = \{z_3, z_4\}.$$

We will show later in the course that a polynomial of degree n always have n complex roots. This is known as the Fundamental Theorem of Algebra.

* <u>Use Mathematica to solve an equation</u>:

One can use the command Solve in Mathematica to find roots of an equation. For example,

```
Solve[z^2 + z + 4 == 0, z]
```

[Note that one needs to write two equal signs.]

To solve for numeric values of roots, one can use the command NSolve.

For example, `NSolve[z^3- 2*z^2 + 2*z - 2 - I == 0, z]`