Lecture 4

Monday, April 6, 2020

A complex number z can be represented in standard form $z = a + ib$ or polar form $z = re^{i\theta}$. One form can be converted to the other. Given the polar form, it is quite easy to get the standard form because

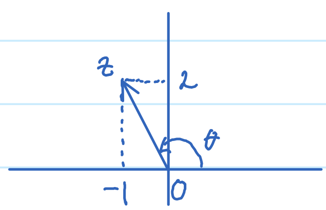
$$\begin{cases} a = r\cos\theta, \\ b = r\sin\theta. \end{cases}$$

Given the standard form, one can find $r = \sqrt{a^2+b^2}$. The argument θ is a little more tricky:

$$\cos\theta = \frac{a}{r} = \frac{a}{\sqrt{a^2+b^2}},$$

$$\sin\theta = \frac{b}{r} = \frac{b}{\sqrt{a^2+b^2}}.$$

Let's consider an example $z = -1 + 2i$.



$$r = |z| = \sqrt{(-1)^2+2^2} = \sqrt{5}.$$

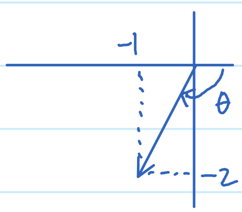
$$\cos\theta = -\frac{1}{\sqrt{5}}, \quad \sin\theta = \frac{2}{\sqrt{5}}.$$

We see that the angle θ lies in between $\pi/2$ and π. The arccos function gives values in the range $[0, \pi]$. Thus,

$$\theta = \arccos\left(-\frac{1}{\sqrt{5}}\right) = 2.0344439\ldots$$

Let's consider another example $w = -1 - 2i$.

$$r = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$$



$$\cos\theta = -\frac{1}{\sqrt{5}}, \quad \sin\theta = -\frac{2}{\sqrt{5}}$$

θ is in the range of $[-\pi, -\pi/2]$ according to the picture. Thus,

$$\theta = -\arccos\left(-\frac{1}{\sqrt{5}}\right) = -2.03444\ldots$$

If $z = re^{i\theta}$ then θ is called an argument of z. It is unique in modulo 2π.

$$\arg(z) = \theta + k2\pi \qquad (k \in \mathbb{Z})$$
$$= \{\theta, \theta+2\pi, \theta-2\pi, \theta+4\pi, \dots\}$$

"arg" is a multi-valued "function". It is not a function is usual sense. It has connection with the logarithm function. We will discuss in detail later.

Arg(z) is the value of argument that lies in $(-\pi, \pi]$. It is called the principal argument of z.

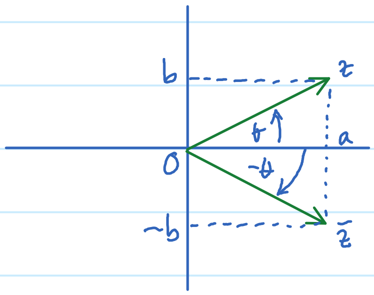
Ex:

$$\arg(-1+2i) = 2.0344\ldots + k2\pi \qquad (k \in \mathbb{Z})$$
$$\mathrm{Arg}(-1+2i) = 2.0344\ldots \quad \in [-\pi, \pi)$$

$$\arg(-1-2i) = -2.0344\ldots + k2\pi \qquad (k \in \mathbb{Z})$$
$$\mathrm{Arg}(-1-2i) = -2.0344\ldots$$

* Complex conjugate:

For $z = a+ib$, the number $\bar{z} = a-ib$ is called is called the complex conjugate of z.



Geometrically, z and $\bar{z}$ are mirror reflection of each other with respect to the x-axis.

$$|z| = |\bar{z}| = r = \sqrt{a^2+b^2}$$

If θ is an argument of z then $-\theta$ is an argument of $\bar{z}$. We also see that

$$z\bar{z} = (a+bi)(a-ib) = a^2+b^2 = |z|^2.$$

* Geometric interpretation of complex multiplication:

Consider two complex numbers written in standard form

$$z = a+ib, \qquad w = c+id.$$

We know how to multiply them using the rule $i^2=-1$.

$$zw = ac-bd+i(ad+bc)$$

The geometric meaning of multiplication is more clear in the polar form. Write $z=re^{i\theta}$ and $w=se^{i\gamma}$. Then

$$zw=(re^{i\theta})(se^{i\gamma}) = rs\, e^{i\theta}e^{i\gamma}.$$

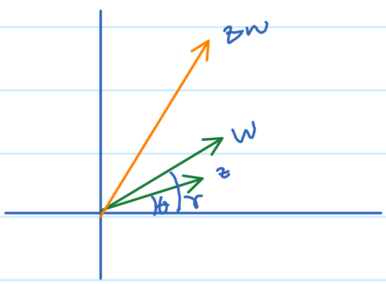
We have

$$\begin{aligned} e^{i\theta}e^{i\gamma} &= (\cos\theta+i\sin\theta)(\cos\gamma+i\sin\gamma) \\ &= (\cos\theta\cos\gamma-\sin\theta\sin\gamma)+i(\sin\theta\cos\gamma+\cos\theta\sin\gamma) \\ &= \cos(\theta+\gamma)+i\sin(\theta+\gamma) \\ &= e^{i(\theta+\gamma)}. \end{aligned}$$

Therefore,

$$zw = rs\, e^{i(\theta+\gamma)}. \qquad (*)$$

The rule of thumb is: to multiply two complex numbers, we multiply their moduli to get the modulus, and add their arguments to get the argument.



Formula (*) has an helpful consequence. Suppose we are to compute $(-1+2i)^{10}$. Of course we can multiply $-1+2i$ by itself ten times, using distribution rule and the rule $i^2=-1$. This practice takes a long time. A more efficient way is as follows

First, we express $z=-1+2i$ in polar form: $z=\sqrt{5}\,e^{i2.0344\ldots}$

Then we raise z to power 10 by raising $r=\sqrt{5}$ by power 10 and multiplying $\theta=2.0344\ldots$ by 10.

$$z^{10} = (\sqrt{5})^{10}e^{i20.344\ldots} = \underbrace{3125e^{i20.344\ldots}}_{\text{polar form of } z^{10}}$$

In general, if n is an integer then

$$(re^{i\theta})^n = r^n e^{in\theta}$$

This is known as de Moivre's formula (~1722).

Now how do we take roots of a complex number? For example, how do we find $\sqrt[3]{-1+2i}$?

Let us put $w = \sqrt[3]{-1+2i}$. We want to solve for complex number w from the equation $w^3 = -1+2i$. Write w in polar form:

$$w = s e^{i\gamma}$$

where s and γ are to be determined. We also write $-1+2i$ in polar form:

$$-1+2i = re^{i\theta}$$

where $r = \sqrt{5}$ and $\theta = 2.0344\ldots$ (as computed earlier). The equation $w^3 = -1+2i$ becomes

$$s^3 e^{i3\gamma} = re^{i\theta}.$$

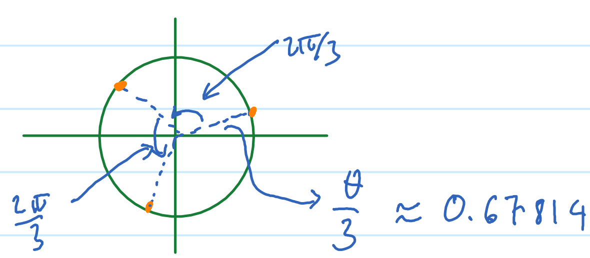
For two complex numbers to be equal to each other, the moduli must be equal and the arguments must be equal in modulo 2π. Thus,

$$\begin{cases} s^3 = r \\ 3\gamma = \theta + k2\pi \quad \text{(for some integer } k\text{)}. \end{cases}$$

We get

$$\begin{cases} s = \sqrt[3]{r} \quad \text{(regular third root of a real number)} \\ \gamma = \dfrac{\theta}{3} + k\dfrac{2\pi}{3}. \end{cases}$$

We see that there are multiple values of w because there are multiple values of γ. Let us put the values of w on the complex plane.



We see that there are only 3 values of w. They are equally spaced on the circle of radius $s = \sqrt[6]{5}$.

In conclution,

$$\sqrt[3]{-1+2i} = \sqrt[6]{5}\, e^{i\left(\frac{\theta}{3}+k\frac{2\pi}{3}\right)} \quad \text{with } k=0,1,2.$$

One can use the same method to find the n'th root of a complex number $z = r e^{i\theta}$.

$$\sqrt[n]{z} = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n}+k\frac{2\pi}{n}\right)} \quad \text{with } k=0,1,2,\dots,n-1.$$