Lecture 3

Friday, April 3, 2020

Last time we defined the set of complex numbers

$$\mathbb{C} = \{a+bi : a, b \in \mathbb{R}\}$$

and the basic arithmetic operations on $\mathbb{C}$. Addition and subtraction are done componentwise:

$$(a+bi) + (c+di) = (a+c) + (b+d)i,$$
$$(a+bi) - (c+di) = (a-c) + (b-d)i.$$

The multiplication and division are somewhat strange, but understandable:

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$
$$\frac{1}{a+bi} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2} i$$

We want to extend the familiar functions of real variables such as the exponential, the trigonometric functions, the logarithm, etc to complex variables. For example, what is e^i?

Taylor series of e^x gives us a hint how to define e^i.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

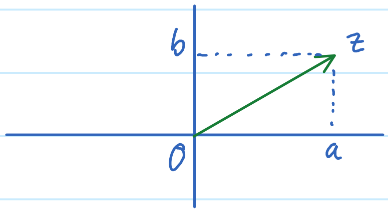
Formally substitute x by i, we get

$$e^i = 1 + i + \frac{i^2}{2!} + \frac{i^3}{3!} + \dots \qquad (*)$$

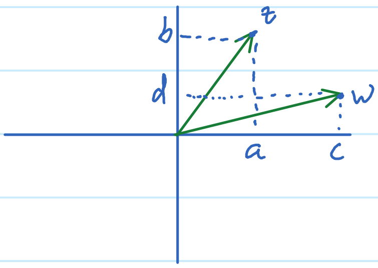
We see that RHS$(*)$ is an infinite sum of complex numbers. We know that infinite sum (series) is the limit of partial sums, so the RHS$(*)$ should be understood as

$$\lim_{n\to\infty} \left(1 + i + \frac{i^2}{2!} + \frac{i^3}{3!} + \dots + \frac{i^n}{n!}\right)$$

We know how to compute the partial sum because we know how to add and multiply two complex numbers. However, we need the notion of limit to make sense of the series. Limit involves to idea of closeness. One needs to be able to tell the distance between two complex numbers. How to define the distance between two complex numbers?



Each complex number $z = a + bi$ corresponds to a vector on the plane (or a point with coordinates (a, b)).



The distance between two complex numbers $z = a + bi$ and $w = c + di$ is defined as the distance between two points (a, b) and (c, d) on the plane. By Pythagorean theorem, this distance is $\sqrt{(a-c)^2 + (b-d)^2}$.

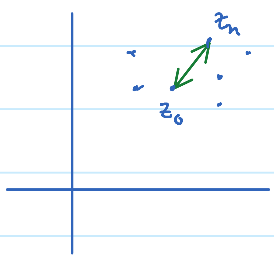
The modulus (or magnitude, or length) of complex number z is the length of the vector that represents z, denoted by $|z|$.

$$|z| = \sqrt{a^2 + b^2}$$

Thus, the distance between z and w is

$$|z - w| = \sqrt{(a-c)^2 + (b-d)^2}$$

With the distance defined above, we can define limit: given a sequence of complex numbers z_n, we say that $\lim_{n\to\infty} z_n = z_0$ if $|z_n - z_0| \to 0$ as $n \to \infty$.



Let us write $z_n = a_n + ib_n$

$z_0 = a_0 + ib_0$

where $a_n, b_n \in \mathbb{R}$. Then

$$|z_n - z_0| = \sqrt{(a_n - a_0)^2 + (b_n - b_0)^2}$$

We see that $|z_n - z_0| \to 0$ if and only if $a_n - a_0 \to 0$ and $b_n - b_0 \to 0$.

In other words, $z_n \to z_0$ if and only if

$$\begin{cases} a_n \to a_0, \\ b_n \to b_0 \end{cases}$$

Thus, the notion of limit of complex numbers is the same as the notion of limit in $\mathbb{R}^2$.

With the notion of limit above, one can define the value of e^i as the limit of the sequence $z_n = 1 + \frac{i}{1!} + \frac{i^2}{2!} + \frac{i^3}{3!} + \cdots + \frac{i^n}{n!}$.

[We will explain why the limit exists another time.]

We have

$$e^i = 1 + \frac{i}{1!} + \frac{i^2}{2!} + \frac{i^3}{3!} + \cdots$$

$$= \left(1 + \frac{i^2}{2!} + \frac{i^4}{4!} + \frac{i^6}{6!} + \cdots\right) + \left(\frac{i}{1!} + \frac{i^3}{3!} + \frac{i^5}{5!} + \frac{i^7}{7!} + \cdots\right)$$

$$= \underbrace{\left(1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots\right)}_{\text{real part}} + i\underbrace{\left(1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots\right)}_{\text{imaginary part}}$$

In general, for $x \in \mathbb{R}$ we can define

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots$$

$$= \left(1 + \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} + \cdots\right) + \left(\frac{ix}{1!} + \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} + \cdots\right)$$

$$= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}_{\cos x} + i\underbrace{\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}_{\sin x}$$

$$= \cos x + i \sin x$$

Def :

$$e^{ix} = \underbrace{\cos x + i\sin x}_{\substack{\text{also denoted} \\ \text{as } cis(x)}} \qquad \forall x \in \mathbb{R}.$$

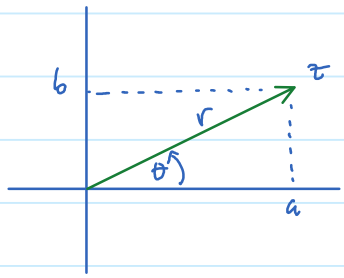
If we take $x = \pi$ then we get $e^{i\pi} = \underbrace{\cos\pi}_{-1} + i\underbrace{\sin\pi}_{0} = -1$. Thus,

$$e^{i\pi} + 1 = 0.$$

This is known as Euler's identity. It is interesting in sense that it involves the 5 important numbers in mathematics, namely 0 (the neutral element of addition), 1 (the neutral element of multiplication), e (the Neper number), π (area of unit circle), i (the imaginary unit).

Polar form of a complex number

A complex number z has standard form $a+bi$ where $a, b \in \mathbb{R}$. It can also be represented in polar form:



Argand's diagram

$$r = |z|$$

$$\cos\theta = \frac{a}{r}, \quad \sin\theta = \frac{b}{r}.$$

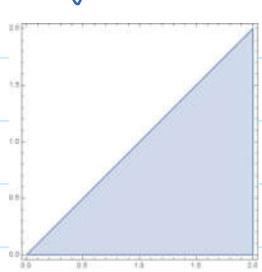
Then

$$\begin{aligned} z &= a + bi \\ &= r\cos\theta + i r\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta}. \end{aligned}$$

The form $z = re^{i\theta}$ is called polar form or (geometric form) of z.
θ is called an argument of z. It is not unique. If θ is an argument then so are $\theta+2\pi$, $\theta-2\pi$, $\theta+4\pi$, ... However, argument is unique in modulo 2π.

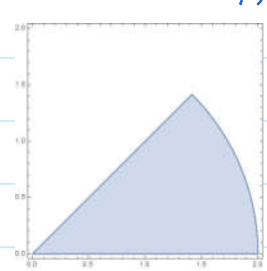
* Examples of ploting regions in Mathematica:

① The region $\{z : 0 \le \theta \le \pi/4\}$.



RegionPlot[0 ≤ Arg[x + I ∗ y] ≤ Pi/4, {x, 0, 2}, {y, 0, 2}]

② The region $\{z : 0 \le \theta \le \pi/4, |z| \le 2\}$.



RegionPlot[0 ≤ Arg[x + I ∗ y] ≤ Pi/4 && Abs[x + I ∗ y] ≤ 2, {x, 0, 2}, {y, 0, 2}]