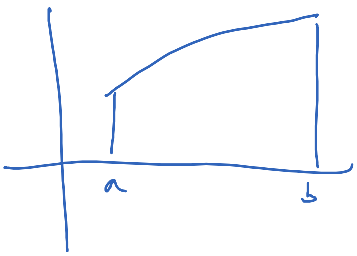
Lecture 28

Friday, June 5, 2020 8:30 AM

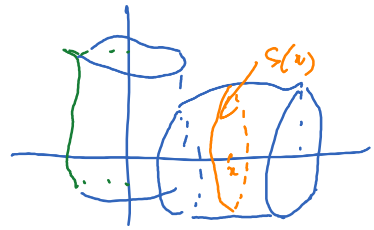
Application of complex integral

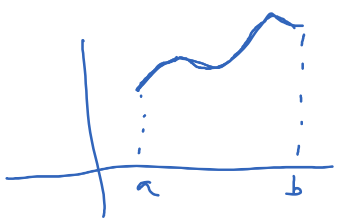
[Office hours on Monday (6/9)
3-4:30 PM
same zoom link.]

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty}\sum_{k=1}^{n-1} f(z_k)(z_{k+1}-z_k)$$



$$\int_a^b f(x)\,dx$$





$$\int_a^b \sqrt{1+f'(x)^2}\,dx$$

$$\int_a^b \underbrace{S(x)}_{\pi f(x)^2}\,dx = \pi\int_a^b f(x)^2\,x$$

$f(z)$ ---- vector field

$f = u+iv$

$z = x+iy$

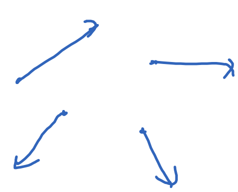
$dz = dx+idy$

$$\underbrace{\int_\gamma f(z)\,dz}$$

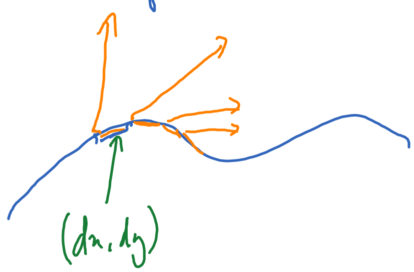
$$= \int_\gamma (u+iv)(dx+idy) = \underbrace{\int_\gamma (u\,dx - v\,dy)}_{?} + i\underbrace{\int_\gamma (v\,dx + u\,dy)}_{?}$$

Recall: $\int_\gamma (P\,dx + Q\,dy) = ?$

$(P(x,y),\ Q(x,y))$



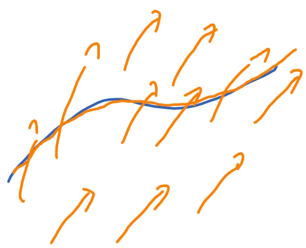
$$\int_\gamma (P\,dx + Q\,dy) = \int_\gamma \underbrace{(P,Q)}\cdot\underbrace{(dx,dy)} = \text{work done by force field } (P,Q) \text{ along } \gamma.$$



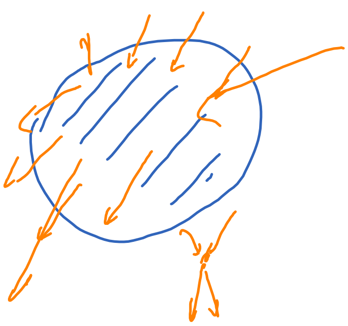
$$\int_\gamma u\,dx - v\,dy \quad \ldots \quad \text{work by } (u,-v) \text{ along } \gamma.$$

$\parallel$ when f is a holo., $\underbrace{(u,-v)}$ is ideal flow.

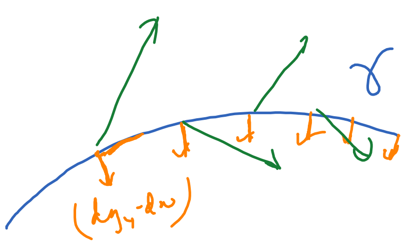
$$\mathrm{Re}\left(\int_\gamma f(z)\,dz\right)$$



$$\mathrm{Im}\left(\int_\gamma f(z)\,dz\right) = \int_\gamma v\,dx + u\,dy$$



$$= \int_\gamma \underbrace{(u,-v)}_{\text{ideal flow}}\cdot\underbrace{(dy,-dx)}_{\text{normal vector}} = \text{flux of ideal flow } (u,-v) \text{ across } \gamma.$$



$$(a,b) \dashrightarrow (b,-a)$$

$$a+ib \qquad \underbrace{b-ia}_{-i(a+ib)}$$

$$= e^{i(-\frac{\pi}{2})}(a+ib)$$

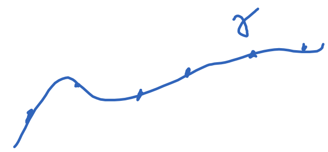
Another application: compute real integral.

Theorem (ML-estimate)

complex function $f(z)$

curve γ, length L

$|f(z)| \le M \quad \forall z \in \gamma$

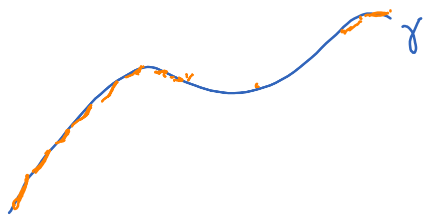


Then

$$\left| \int_\gamma f(z)\, dz \right| \le ML$$

$$\int_\gamma f(z)dz = \lim \underbrace{\sum f(z_k)(z_{k+1} - z_k)}$$

$$|\ldots| \le \sum_{k=0}^{n-1} \underbrace{|f(z_k)|}_{\le M} |z_{k+1} - z_k|$$



$$\le M \underbrace{\sum_{k=0}^{n-1} |z_{k+1} - z_k|}_{\sim L}$$

Ex: $\displaystyle\int_0^\infty \frac{1}{\underbrace{1+x^4}_{\text{even}}} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx$

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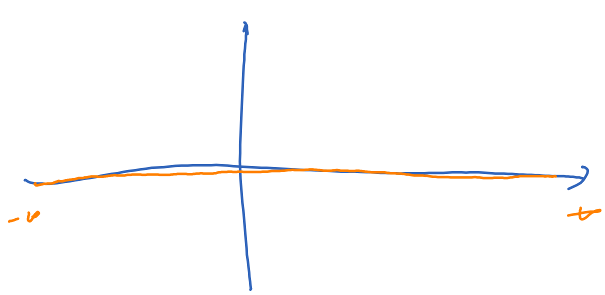
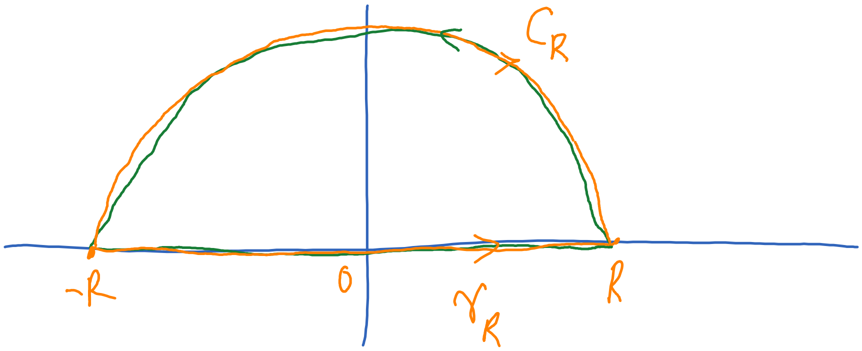
$$= \frac{1}{2} \int_\gamma \frac{1}{1+z^4} dz$$





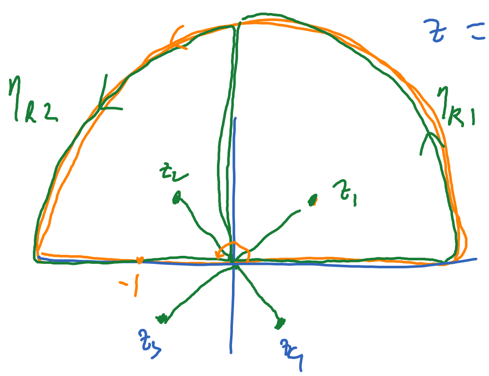
$$\int_\gamma \frac{1}{1+z^4}\,dz = \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{1+z^4}\,dz$$

$$\gamma_R - C_R = \eta_R$$

$$\int_{\eta_R} \underbrace{\frac{1}{1+z^4}}_{f(z)}\,dz =$$

$$1+z^4 = 0 \rightsquigarrow z^4 = -1$$

$$z = \sqrt[4]{-1} = \cdots$$



$$z_1 = e^{i\frac{\pi}{4}}$$

$$z_2 = e^{i\frac{3\pi}{4}}$$

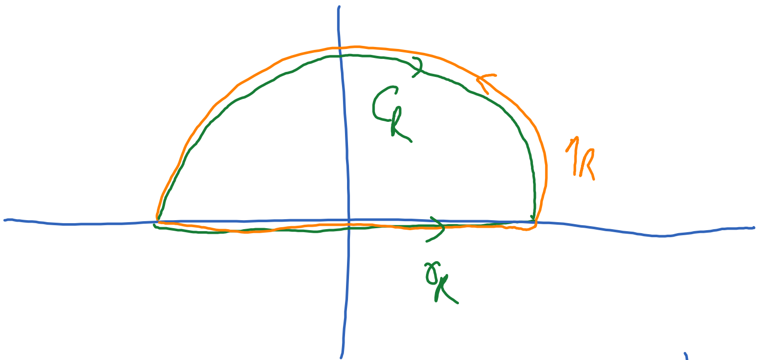
$$z_3 = e^{i\frac{5\pi}{4}}$$

$$z_4 = e^{i\frac{7\pi}{4}}$$

$$f(z) = \frac{1}{z^4+1} = \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

$$\int_{\eta_R} f(z)\,dz = \int_{\eta_R} \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}\,dz = \int_{\eta_{R1}} + \int_{\eta_{R2}}$$

$$\frac{\frac{1}{(z-z_2)(z-z_3)(z-z_4)}}{z-z_1} \qquad \frac{\frac{1}{\cdots}}{z-z_2}$$



$$\underbrace{\int_{\eta_R}}_{\text{known}} = \underbrace{\int_{\gamma_R}}_{\text{want}} - \underbrace{\int_{C_R}}_{\xrightarrow{R\to+\infty} 0}$$

$$\left| \int_{C_R} \frac{1}{z^4+1}\, dz \right| \leq ML \sim \frac{1}{R^4}\pi R \xrightarrow[R\to+\infty]{} 0$$

$$L = \pi R$$

$$\left| \frac{1}{z^4+1} \right| = \frac{1}{|z^4+1|} \sim \frac{1}{R^4} \qquad \text{when } \underbrace{z \in C_R}_{|z| = R}$$

$$\int_{\gamma} \frac{1}{1+z^4}\, dz = \sqrt{2}\,\frac{\pi}{4} \ ??$$

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