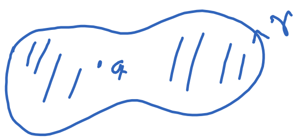
Lecture 27

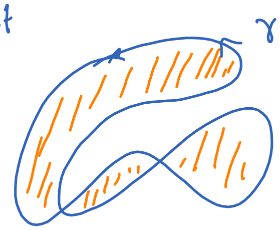
Wednesday, June 3, 2020 8:55 AM

Cauchy's Int form. $\displaystyle\int_\gamma \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$ ✓

- γ pos. oriented
- γ loop, simple
- γ encloses a
- f is holo inside γ



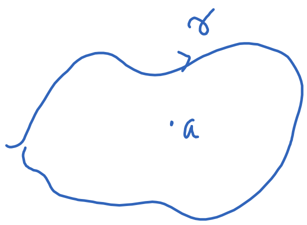
Cauchy-Goursat



- γ loop
- f is holo "inside" γ

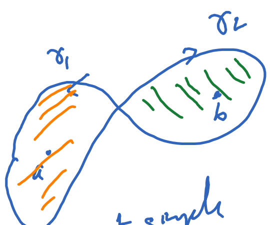
Then $\displaystyle\int_\gamma f(z)\,dz = 0$

Variations of Cauchy's Int form.



- γ neg. oriented
- simple

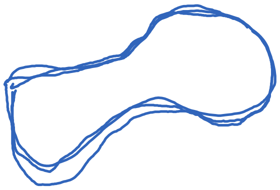
$$\int_\gamma \frac{f(z)}{z-a}\,dz = -2\pi i f(a)$$



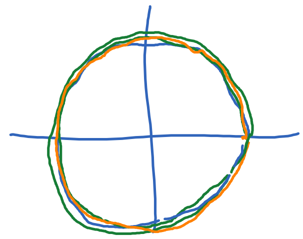
not simple

$$\int_\gamma \frac{f(z)}{z-a}\,dz = \underbrace{\int_{\gamma_1}}_{2\pi i f(a)} + \underbrace{\int_{\gamma_2}}_{=0}$$

$$\int_\gamma \frac{f(z)}{z-b}\,dz = \underbrace{\int_{\gamma_1}}_{=0\ (C-G)} + \underbrace{\int_{\gamma_2}}_{-2\pi i f(b)}$$



$\gamma(t) = e^{it}$ $\qquad 0 \le t \le 8\pi$



$t = 2\pi$ }
$t = 4\pi$ }
$t = 6\pi$ }
$t = 8\pi$

$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$

$f(z) = 1$

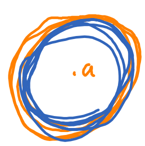
$$\int_\gamma \frac{1}{z-0}\,dz = \int_{\gamma_1} \frac{1}{z}\,dz + \dots + \int_{\gamma_4} \frac{1}{z}\,dz = 4\int_{\gamma_1} \frac{1}{z}\,dz$$

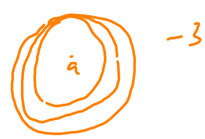
$$= 4\left(2\pi i f(0)\right)$$

$$= 8\pi i$$

$$\int_\gamma \frac{f(z)}{z-a}\,dz = n\, 2\pi i f(a)$$

winding number

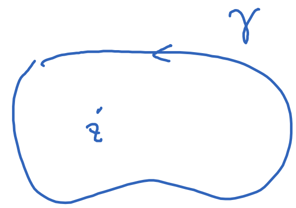




$$\left[\int_\gamma \frac{f(z)}{z-a}\,dz = 2\pi i f(a)\right] \xrightarrow[a \to z]{z \to w} \int_\gamma \frac{f(w)}{w-z}\,dw = 2\pi i f(z)$$

$$\int_\gamma \frac{f(z)}{(z-a)^2}\,dz = ??$$

$$\int_\gamma \frac{f(w)}{w-z}\,dw = 2\pi i f(z)$$



$\int$ f is holo inside γ

$$\frac{d}{dz}\int_\gamma \frac{f(w)}{w-z}\,dw = 2\pi i f'(z)$$

$$\int_\gamma \underbrace{\frac{d}{dz}\frac{f(w)}{w-z}}_{\frac{f(w)}{(w-z)^2}}\,dw = 2\pi i f'(z)$$

$$\int_\gamma \frac{f(w)}{(w-z)^2}\,dw = 2\pi i f'(z)$$

$$\int_\gamma \underbrace{\frac{d}{dz}\frac{f(w)}{(w-z)^2}}_{2\frac{f(w)}{(w-z)^3}}\,dw = 2\pi i f''(z)$$

$$\int_\gamma \frac{f(w)}{(w-z)^3}\,dw = \frac{2\pi i}{2} f''(z)$$

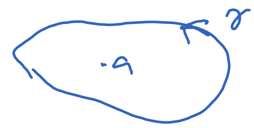
$$\int_\gamma \frac{f(w)}{(w-z)^{n+1}}\,dw = \frac{2\pi i}{n!} f^{(n)}(z)$$

$w \to z$

$z \to a$

$$\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

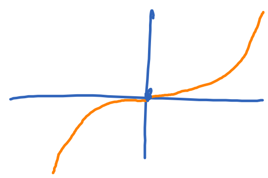
General Cauchy Integral formula



If f is holo. in G then it is inf'ly holomorphic

$f', f'', f''', \ldots$

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

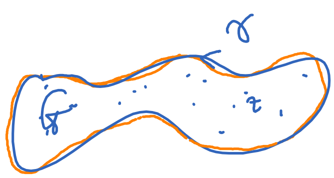


f'

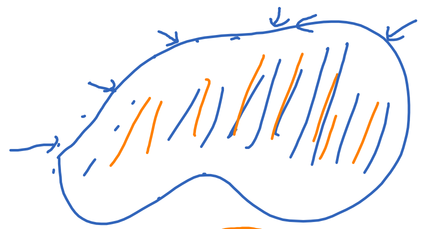
f''

$$\int_\gamma \frac{f(w)}{w-z}\,dw = 2\pi i f(z)$$

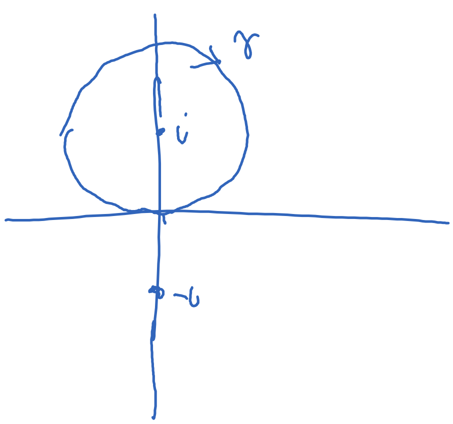
f in G is completely determined by f on γ.



f is holo



Ex:

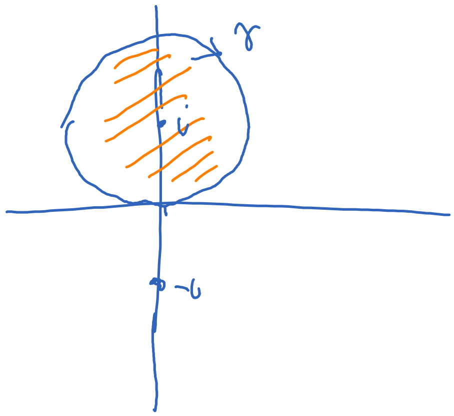


$$\int_\gamma \frac{1}{(z^2+1)^2}\,dz = ?$$

$$(z^2+1)^2 = (z^2 - i^2)^2$$

$$= (z-i)^2(z+i)^2$$

$\pm i$



$$\int_\gamma \frac{1}{(z^2+1)^2}\,dz = ?$$

$$\int_\gamma \frac{1}{\underbrace{(z-i)^2}\underbrace{(z+i)^2}}\,dz = \int_\gamma \frac{\frac{1}{(z+i)^2}}{(z-i)^2}\,dz \qquad \frac{1}{(z+i)^2} = g(z)$$

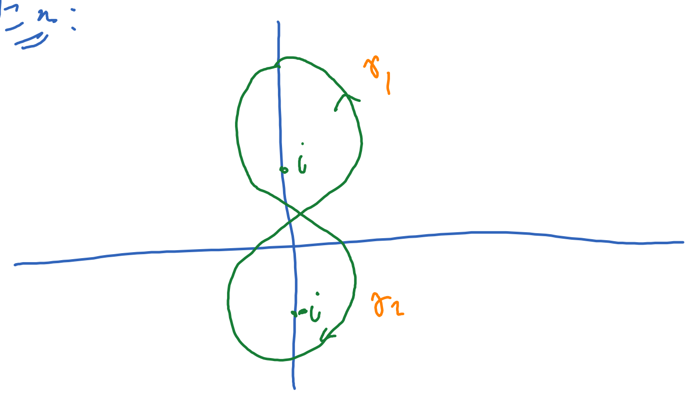
g is holo. inside γ

$$\int_\gamma \frac{g(z)}{(z-i)^2}\,dz = \frac{2\pi i}{1!}\,g'(i)$$

$$g'(z) = \frac{-2}{(z+i)^3}$$

$$g'(i) = \frac{-2}{(i+i)^3} = \dots$$

Ex:



$$\int_\gamma \frac{1}{(z^2+1)^2}\,dz = ?$$

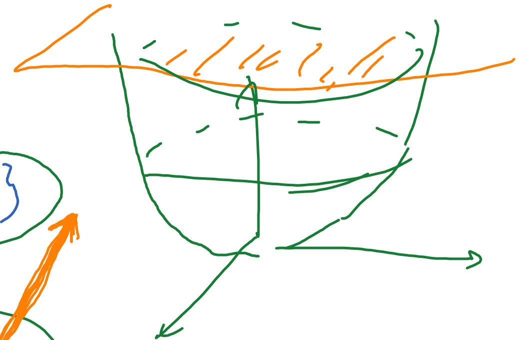
$$= \int_{\gamma_1} + \int_{\gamma_2}$$

$$\underbrace{\int_{\gamma_1} \frac{\frac{1}{(z+i)^2}}{(z-i)^2}\,dz}_{=\frac{2\pi i}{1!}g'(i)} + \underbrace{\int_{\gamma_2} \frac{\frac{1}{(z-i)^2}}{(z+i)^2}}_{= -\frac{2\pi i}{1!}h'(-i)}$$

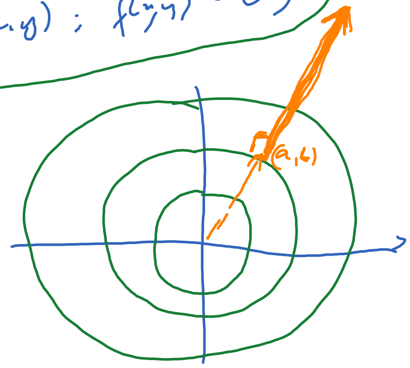
$$\frac{1}{(z+i)^2} = g(z) \qquad \frac{1}{(z-i)^2} = h(z)$$

$f(x,y) = x^2 + y^2$

C- Level set of f



$\{(x,y) : f(x,y) = C\}$



$\nabla f = (f_x, f_y)$
$= (2x, 2y)$
$= 2(x,y)$

$\nabla \psi \perp$ vector field

$\nabla \psi(x,y) \perp$ vector field at (x,y)