Lecture 26

Monday, June 1, 2020 8:17 AM

Cauchy Integral formula:

γ simple loop

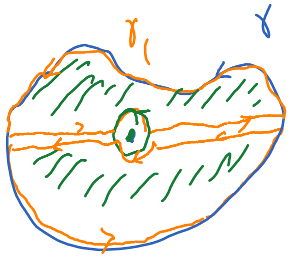
positively oriented

a is enclosed in γ

f holomorphic in G

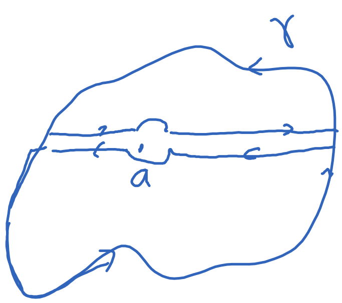
$$\left[\text{Then} \quad \int_\gamma \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a).\right] \qquad (*)$$

Last time



$$\int_\gamma \frac{1}{z}\,dz = 2\pi i \qquad ???$$

$$\int_\gamma = \int_{\gamma_1}$$



simple loop

pos. oriented

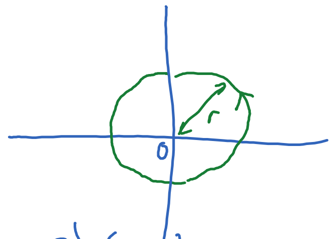
$$\int_\gamma \frac{1}{z-a}\,dz = 2\pi i$$

$$\int_\gamma \frac{f(z)}{z-a}\,dz = \int_\gamma \left(\frac{f(z)-f(a)}{z-a} + \frac{f(a)}{z-a}\right)dz$$

$$= \underbrace{\int_\gamma \frac{f(z)-f(a)}{z-a}\,dz}_{\text{Cauchy - Goursat thm}} + \underbrace{\int_\gamma \frac{f(a)}{z-a}\,dz}_{f(a)\underbrace{\int_\gamma \frac{1}{z-a}dz}_{2\pi i}}$$

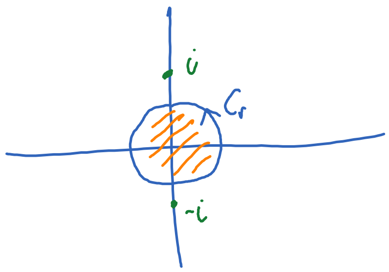
$= 0$

Ex: $\displaystyle\int_{C_r} \frac{1}{z^2+1}\,dz$



$f(z) = \dfrac{1}{z^2+1}$ is holo. in $\mathbb{C}\setminus\{\pm i\}$

If $r<1$:

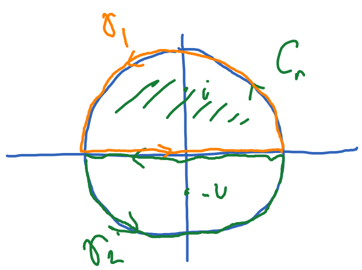


f is holo. inside C_r

$$\int_{C_r} f(z)\,dz = 0$$

Cauchy-Goursat thm

If $r>1$.



$$\int_{C_r} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

$$\underbrace{\int_{\gamma_1}}_{\int_{\frown} + \int_{\rightarrow}} + \underbrace{\int_{\gamma_2}}_{\int_{\smile} + \int_{\leftarrow}}$$

$$\int_{\frown} + \int_{\smile} = \int_{C_r}, \qquad \int_{\rightarrow} + \int_{\leftarrow} = 0$$

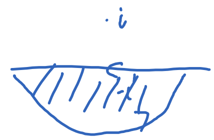
$$\int_{\gamma_1} \frac{1}{z^2+1}\,dz = \int_{\gamma_1} \frac{1}{z^2-i^2}\,dz = \int_{\gamma_1} \frac{1}{(z-i)(z+i)}\,dz$$

$$= \int_{\gamma_1} \frac{\overset{g(z)}{\frac{1}{z+i}}}{z-i}\,dz = 2\pi i\, g(i)$$

$$= 2\pi i \frac{1}{i+i} = \pi$$

g is holo inside γ_1.

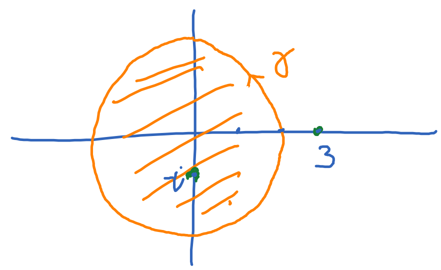
$$\int_{\gamma_2} \frac{1}{z^2+1}\,dz = \int_{\gamma_2} \frac{\overbrace{\frac{1}{z-i}}^{h(z)}}{z+i}\,dz = 2\pi i\, h(-i) = \dots$$



Part (c) of worksheet.

Ex:

$$\int_\gamma \frac{z+1}{(z-3)(z+i)}\,dz$$



$$= \int_\gamma \frac{\overbrace{\frac{z+1}{z-3}}^{g(z)}}{z-\underbrace{(-i)}_{a}}\,dz = 2\pi i\, g(-i) = \dots$$

Ex:

$$\int_\gamma \frac{z}{z^2-1}\,dz = \int_{\gamma_1} + \int_{\gamma_2}$$

$$\gamma(t) = 1+i+2\cos t + 2i\sin t = 1+i+2(\underbrace{\cos t + i\sin t}_{e^{it}})$$

Mathematica: $\gamma(t) = \underbrace{1+2\cos t}_{x(t)} + i(\underbrace{1+2\sin t}_{y(t)})$

p[s_] := ParametricPlot[{ , }, {t, 0, s}]

$0.1 \to 4\pi$

$$\gamma(t) = 1+i+\underbrace{2e^{it}}$$

$0 \le t \le 4\pi$

$0 \le t \le 2\pi$

$2\pi \le t \le 4\pi$