Lecture 25

Friday, May 29, 2020 8:53 AM

$$\int_\gamma z^i \, dz = ?$$

$$z^i = e^{i \operatorname{Log} z}$$

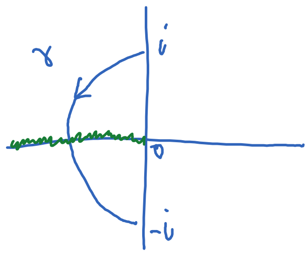
$$z^a \rightsquigarrow \frac{z^{a+1}}{a+1} = \underbrace{\frac{1}{a+1} e^{(a+1)\operatorname{Log} z}}$$

principal branch of Logarithm

$$\frac{d}{dz} \ldots = \frac{1}{z} \underbrace{e^{(a+1)\operatorname{Log} z}}$$

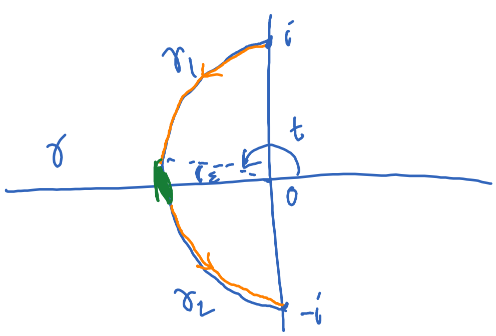
$$= \frac{1}{z} z^{a+1}$$

$$= \frac{1}{\cancel{z}} z^a \cancel{z} = z^a$$



$\frac{z^{a+1}}{a+1}$ is an antid. of z^a on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

z^i has an anti. $\frac{z^{i+1}}{i+1}$ on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.



$$\gamma(t) = e^{it} \qquad \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$$

$$\gamma_1(t) = e^{it} \qquad \frac{\pi}{2} \leq t \leq \pi - \varepsilon$$

$$\gamma_2(t) = e^{it} \qquad \pi + \varepsilon \leq t \leq \frac{3\pi}{2}$$

$$\int_\gamma z^i \, dz = \lim_{\varepsilon \to 0} \left(\int_{\gamma_1} z^i \, dz + \int_{\gamma_2} z^i \, dz \right)$$

Fund. Thm of Calculus.

$\int_\gamma f(z)\,dz$ ----- $\lim_{n\to\infty} \underbrace{\sum_{k=0}^{n-1} f(z_k)(z_{k+1}-z_k)}_{\text{slow to converge}}$

$f\left(\frac{z_k+z_{k+1}}{2}\right)$

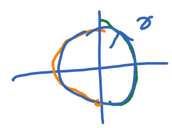
$\int_a^b f(\gamma(t))\,\underline{\gamma'(t)}\,dt$

Fund. thm

$$\boxed{\begin{array}{l} f(z) \text{ has antid. } F(z) \\ \int_\gamma f(z)\,dz = F(B)-F(A). \end{array}}$$

$f(z) = \frac{1}{z}$

$F(z) = \log z$



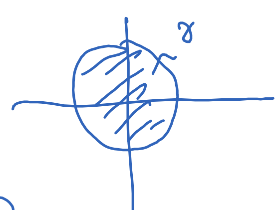
$\gamma(t) = e^{it}$

$\int_\gamma \frac{1}{z}\,dz = 2\pi i$

Cauchy-Goursat thm.

$\int_\gamma f(z)\,dz = 0$ if f is <u>holo.</u> in region G enclosed by γ.

Ex: $\int_\gamma \underbrace{\left(\frac{1}{z^2}+\frac{1}{z^3}\right)}_{f(z)} dz$



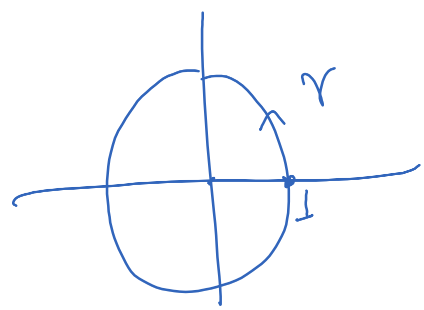
Does f have an antid. in some region G that contains γ?

f is not holo. in D

$$\frac{1}{z^2} = z^{-2} \quad \cdots\cdots \quad -\frac{1}{z} \xrightarrow{\text{holo}} \mathbb{C}\setminus\{0\}$$

$$\frac{1}{z^3} = z^{-3} \quad \cdots\cdots \quad \frac{1}{-2} z^{-2} = \frac{-1}{2z^2} \xrightarrow{\text{holo}}$$

$$\underbrace{\frac{1}{z^2} + \frac{1}{z^3}}_{f(z)} \quad \text{has anti:} \quad \frac{-1}{z} + \frac{-1}{2z^2} \quad \text{in } \mathbb{C}\setminus\{0\}.$$

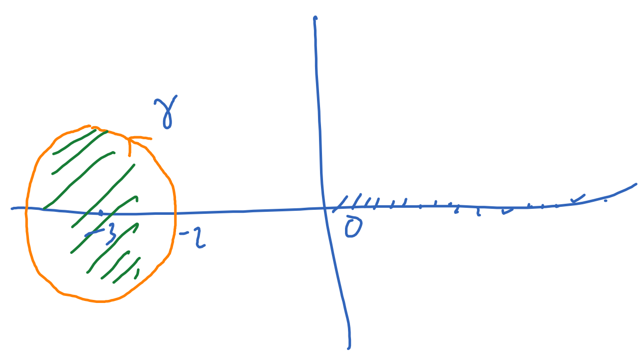


$\gamma \subset \mathbb{C}\setminus\{0\}$

$$\int_\gamma f(z)\,dz = F(1) - F(1) = 0$$

Ex: $\displaystyle\int_\gamma \frac{z+1}{z}\,dz = 0$

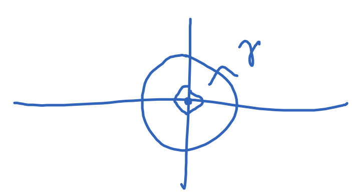
by Cauchy-Goursat theorem.



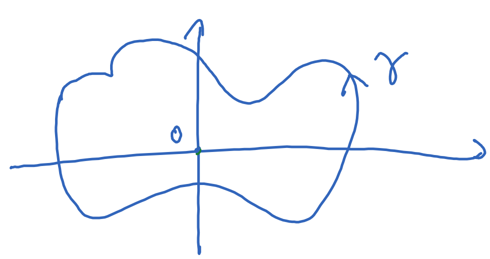
$1 + \frac{1}{z}$

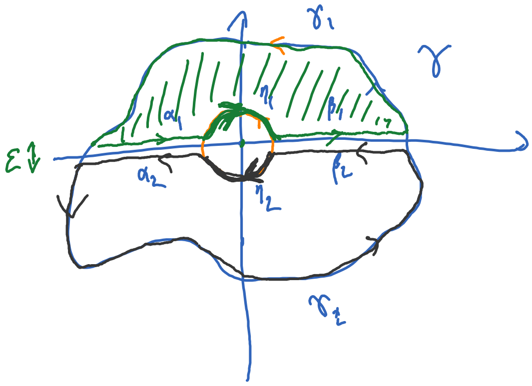
$F(z) = z + \mathrm{Log}(-z)$

Ex $\displaystyle\int_\gamma \frac{1}{z}\,dz = 2\pi i$



γ simple loop





$\frac{1}{z}$ has no antid. in $\mathbb{C}\setminus\{0\}$.

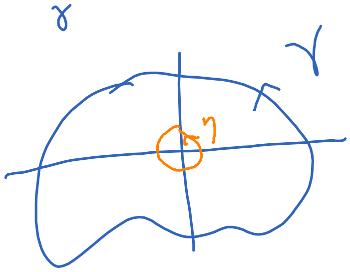
$$\underbrace{\int_{\gamma_1+\alpha_1+\eta_1+\beta_1}}_{=0} + \underbrace{\int_{\gamma_2+\beta_2+\eta_2+\alpha_2}}_{=0} = 0$$

$\frac{1}{z}$ is holo. in the region enclosed by $\gamma_1+\alpha_1+\eta_1+\beta_1$.

$$0 = \underbrace{\left(\int_{\gamma_1} + \int_{\gamma_2}\right)}_{\int_\gamma} + \underbrace{\left(\int_{\alpha_1} + \int_{\alpha_2}\right)}_{0} + \underbrace{\left(\int_{\beta_1} + \int_{\beta_2}\right)}_{0} + \underbrace{\left(\int_{\eta_1} + \int_{\eta_2}\right)}_{\int_{-\eta}}$$

$\downarrow$

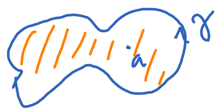
0



$$0 = \int_\gamma \frac{1}{z}\,dz + \int_{-\eta} \frac{1}{z}\,dz$$

$$\int_\gamma \frac{1}{z}\,dz = \int_\eta \frac{1}{z}\,dz = 2\pi i$$

Thm (Cauchy's Integral formula)



$\left\{\begin{array}{l} \gamma \text{ encloses } a \\ f(z) \text{ holo. in } G \text{ (enclosed by } \gamma) \\ \gamma \text{ is simple, positively oriented} \end{array}\right.$

Then $\displaystyle\int_\gamma \frac{f(z)}{z-a}\,dz = 2\pi i f(a).$