Lecture 24

Wednesday, May 27, 2020 8:52 AM

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

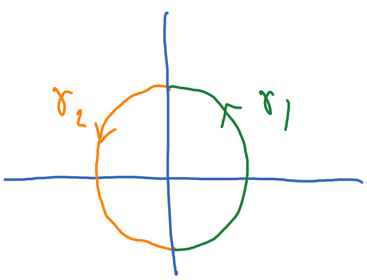
If f has antider. F in region $\underset{\gamma \subset G}{\underbrace{G}}$ then

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

$f(z) = \frac{1}{z}$

$\int_\gamma \frac{1}{z}\,dz \longrightarrow$ one can't use Fund Thm of Calc.

$\frac{1}{z}$ doesn't have antider. on $\mathbb{C}\setminus\{0\}$.



$$\int_\gamma \frac{1}{z}\,dz = \int_{\gamma_1} \frac{1}{z}\,dz + \int_{\gamma_2} \frac{1}{z}\,dz$$

$\gamma_1 \subset \underbrace{\mathbb{C}\setminus\mathbb{R}_{\leq 0}}$ $\frac{1}{z}$ has antid $\underline{\operatorname{Log} z}$

$\gamma_2 \subset \underbrace{\mathbb{C}\setminus\mathbb{R}_{\geq 0}}$ $\frac{1}{z}$ has anti $\underline{\operatorname{Log}(-z)}$

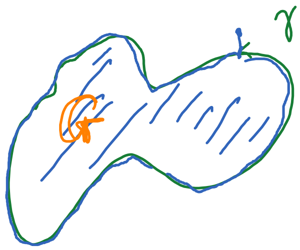
$\frac{1}{z} \ \ldots \ \underbrace{\operatorname{Log} z}_{G = \mathbb{C}\setminus\mathbb{R}_{\leq 0}}$

$\frac{1}{z} \ \ldots \ \underbrace{\operatorname{Log}(-z)}_{G = \mathbb{C}\setminus\mathbb{R}_{\geq 0}}$

$\underbrace{\operatorname{Log}(e^{i\frac{\pi}{3}} z)}$

$e^{i\frac{\pi}{3}} z \in \mathbb{R}_{\leq 0}$

Cauchy - Goursat theorem

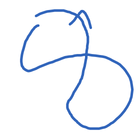


γ $\underbrace{\text{closed curve}}_{\text{loop}}$

γ is $\underline{\text{simple curve}}$

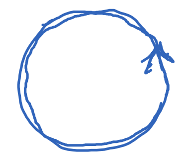
doesn't intersect itself

except at the end points



not simple

γ is $\underbrace{\text{positive oriented}}$



G is the region enclosed by γ.

* Consider a function $f(z)$ holomorphic in G

$$\int_\gamma f(z)\,dz$$

Cauchy-Goursat thm: $\displaystyle\int_\gamma f(z)\,dz = 0$

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_a^b \big(u(\underset{x(t)}{x},\underset{}{y}) + i\,v(x,\underset{y(t)}{y})\big)\big(x'(t) + i y'(t)\big)\,dt$$

$$\gamma(t) = x(t) + i y(t)$$

$$\gamma'(t) = x'(t) + i y'(t)$$

$$f(z) = u(x,y) + i v(x,y)$$

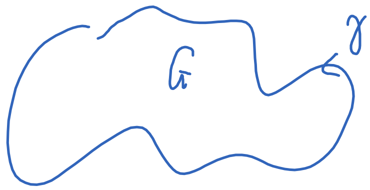
$$= \int_a^b \Big[u(x,y)\,x'(t) - v(x,y)\,y'(t) + i\big(u(x,y)\,y'(t) + v(x,y)\,x'(t)\big)\Big]dt$$

$$= \int_a^b \big(u(x,y)\,x'(t) - v(x,y)\,y'(t)\big)\,dt + i\int_a^b \big(u(x,y)\,y'(t) + v(x,y)\,x'(t)\big)\,dt$$

$$\int_\gamma f(z)\,dz = \underbrace{\int_\gamma \big(u(x,y)\,dx - v(x,y)\,dy\big)}_{\text{line int}} + i\underbrace{\int_\gamma \big(u(x,y)\,dy + v(x,y)\,dx\big)}_{\text{line int}}$$

$$\int_\gamma f(z)\,dz = \underbrace{\int_\gamma (u\,dx - v\,dy)}_{A\,=\,0} + i\underbrace{\int_\gamma (u\,dy + v\,dx)}_{B\,=\,0}$$

Green's theorem



$$\int_\gamma (P\,dx + Q\,dy) = \iint_G (Q_x - P_y)\,dA$$

$$\underbrace{Q_x - P_y} = \mathrm{curl}(P, Q)$$

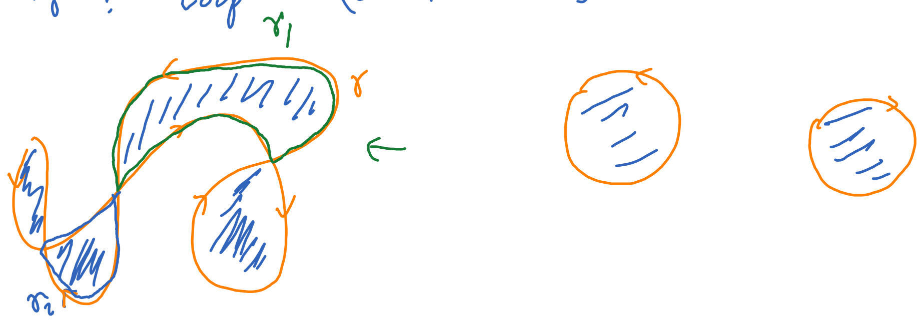
$$A = \int_\gamma P\,dx + Q\,dy \qquad P = u, \quad Q = -v$$

$$= \iint_G (Q_x - P_y)\,dA = \iint_G (\underbrace{-v_x - u_y}_{=0})\,dA = 0.$$

$=0$ because f is holo. in G.

Thm (Cauchy-Goursat)

γ : loop (closed curve)



$$\int_\gamma f\,dz = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4}$$

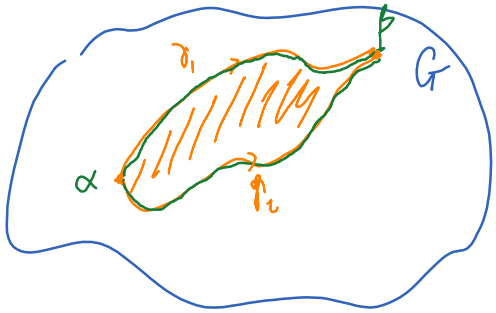
f is holo. in G enclosed by γ.

Then $\int_\gamma f(z)\,dz = 0$.

$$\int_\gamma f(z)\,dz = -\underbrace{\int_{-\gamma} f(z)\,dz}_{=0}$$

$$\int_a^a f(x)\,dx = 0$$

Consequences



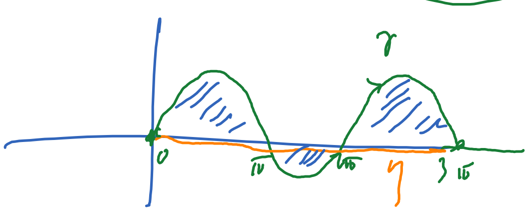
f is holo. in G.

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

$$\underbrace{\gamma = \gamma_1 - \gamma_2} \qquad \underbrace{\int_{\gamma} f(z)\,dz = 0}_{\int_{\gamma_1} - \int_{\gamma_2}}$$

$$\int_{\gamma} e^z\,dz = e^z \Big|_0^{3\pi}$$ dangerous

e^z has <u>anti</u> e^z in $\mathbb{C}$.



$$\int_{\gamma} e^z\,dz = e^{3\pi} - e^0$$

e^z is holo everywhere.

Cauchy Goursat: $$\int_{\gamma} e^z\,dz = \int_{\eta} e^z\,dz = \int_0^{3\pi} e^x\,dx = e^x\Big|_0^{3\pi} = e^{3\pi} - 1.$$