Lecture 23

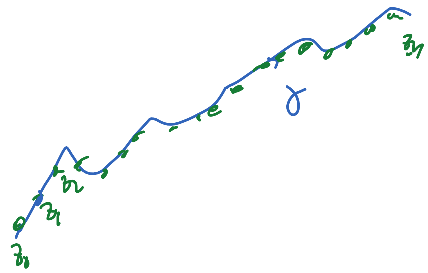
Friday, May 22, 2020 8:56 AM

$\int_\gamma f(z)\,dz \longrightarrow$ def. $\underbrace{\lim_{n\to\infty} \sum_{k=0}^{n-1} f(z_k)(z_{k+1}-z_k)}$

$\gamma(t)$

$\int_a^b f(\gamma(t))\,\gamma'(t)\,dt$

Fundam. Th. of Calc.



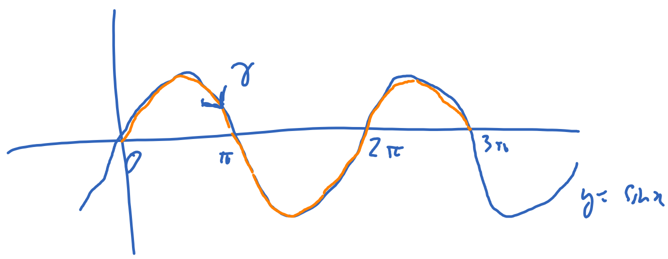
$z = \gamma(t) \qquad a \le t \le b$

$dz = \gamma'(t)\,dt$

Calc I $\left[\left[\, y = f(x) \quad dy = f'(x)\,dx \,\right]\right]$

Ex:

$\int_\gamma e^z\,dz$



$= \int_0^{3\pi} e^{\gamma(t)}\,\gamma'(t)\,dt$

$= \int_0^{3\pi} \underbrace{e^{t+i\sin t}(1+i\cos t)}_{g(t)}\,dt$

$\begin{cases} x = t \\ y = \sin t \end{cases} \qquad 0 \le t \le 3\pi$

$\gamma(t) = x(t) + i\,y(t)$

$\gamma(t) = t + i\sin t$

$\gamma'(t) = 1 + i\cos t$

$= \int_0^{3\pi} \left(e^t\cos(\sin t) + i\,e^t\sin(\sin t) \right) \times (1+i\cos t)\,dt$

$\approx \sum_{k=0}^{n-1} g(t_k)\,\underbrace{(t_{k+1}-t_k)}$

$\rightsquigarrow$ complicated!

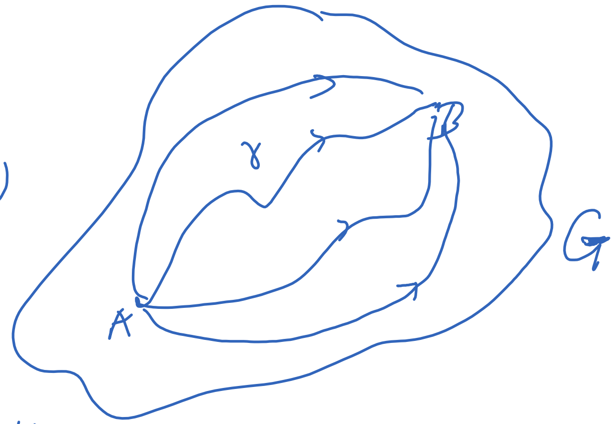
Thm (Fundamental Thm of Cal for complex variable)

$f(z)$

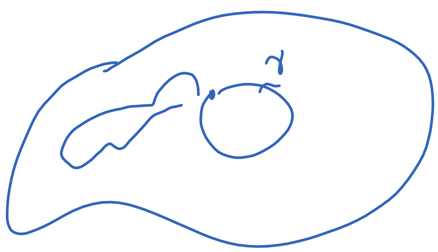
Suppose F is an antiderivative of f in region G.

Suppose γ lies entirely in G

Then
$$\int_\gamma f(z)\,dz = F(B) - F(A)$$



Corollary: If γ is a loop in G then
$$\int_\gamma f(z)\,dz = F(B) - F(A) = 0.$$



$$F' = f$$

Why?
$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_a^b \underbrace{F'(\gamma(t))\,\gamma'(t)}\,dt \longrightarrow \text{chain rule}$$

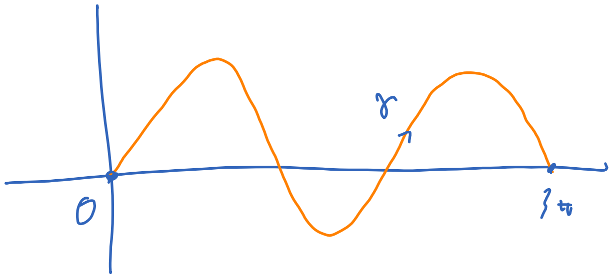
$$f(z) = F'(z)$$

$$= \int_a^b [F(\gamma(t))]'\,dt$$

$$= F(\gamma(b)) - F(\gamma(a))$$

} Fund thm of calc (real variable).

$$= F(B) - F(A)$$



$f(z) = e^z$

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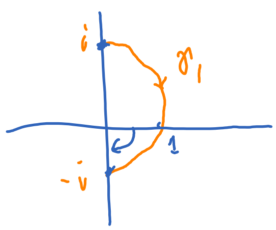
$$\int_\gamma f(z)\,dz = F(3\pi) - F(0) = e^{3\pi} - e^0 = e^{3\pi} - 1$$

$$\approx 12390.6\ldots$$

(a) :

$$\int_{\gamma_1} \frac{1}{z}\,dz = \int_{-\pi/2}^{\pi/2} \frac{1}{\gamma_1(t)}\,\gamma_1'(t)\,dt = \int_{-\pi/2}^{\pi/2} \frac{1}{\cancel{e^{-it}}}(-i)\cancel{e^{-it}}\,dt$$



$\gamma_1(t) = e^{-it}$

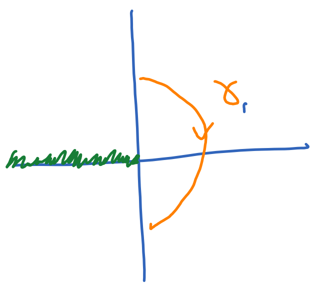
$-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

$$= -i \int_{-\pi/2}^{\pi/2} dt$$

$$= -i\pi$$

$\gamma_1'(t) = \underset{\sim}{-i}\,e^{-it}$

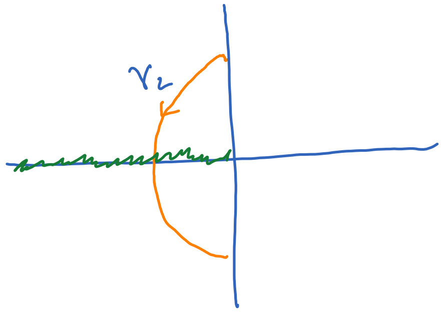
$\frac{1}{z}$ . . . . $\operatorname{Log} z$ is an antid. of $\frac{1}{z}$ in $\underbrace{\mathbb{C} \setminus \mathbb{R}_{\leq 0}}_{G}$



G

$$\int_{\gamma_1} \frac{1}{z}\,dz = \operatorname{Log}(-i) - \operatorname{Log}(i) = \ldots$$

(b)



$$\int_{\gamma_2} \frac{1}{z}\, dz$$

Parameter of γ_2: $\displaystyle\int \frac{1}{\gamma_2(t)}\, \gamma_2'(t)\, dt$

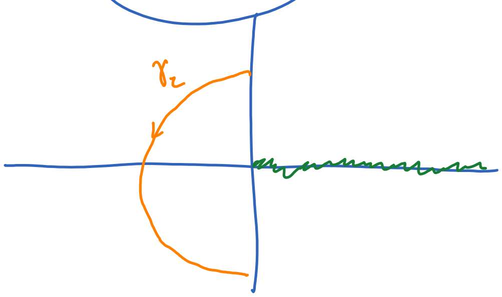
$\frac{1}{z}$ has anti $\underbrace{\text{Log } z}_{\mathbb{C} \setminus \mathbb{R}_{\leq 0}}$

$$\left(\text{Log}(az)\right)' = a \frac{1}{az} = \frac{1}{z}.$$

$\text{Log}(-z)$ is also anti of $\frac{1}{z}$.

$-z \in \mathbb{R}_{\leq 0}$

$z \in \mathbb{R}_{\geq 0}$



$$\underbrace{\int_{\gamma_2} \frac{1}{z}\, dz}_{} = \text{Log}(-z) \Big|_{i}^{-i} = \text{Log}(i) - \text{Log}(-i).$$

$$\neq \underbrace{\int_{\gamma_1} \frac{1}{z}\, dz}_{}$$