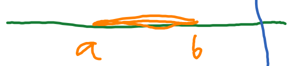
Lecture 22

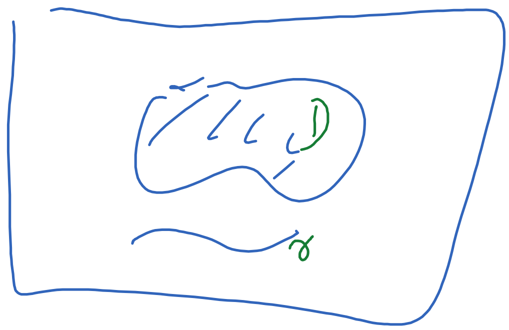
Wednesday, May 20, 2020 9:00 AM

$\mathbb{R} \to \mathbb{R}$ $\qquad\qquad\qquad\qquad$ $\mathbb{C} \to \mathbb{C}$

$$\int_a^b f(x)\,dx \longleftrightarrow \int_\gamma f(z)\,dz$$

$x_{k+1} - x_k$ $\qquad\qquad$ $z_{k+1} - z_k$

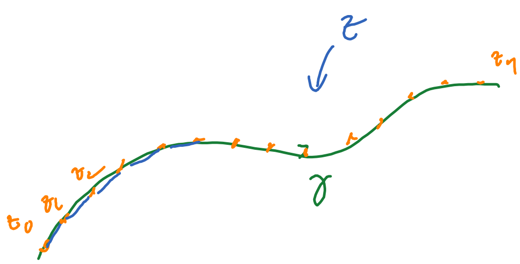




$$\iint_D f(x,y)\,dx\,dy$$

$$\cancel{\int_D f(z)\,dz}$$

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(z_k)\,(z_{k+1} - z_k)$$



$\gamma(t)$

$$\boxed{z_k = \gamma(t_k)}$$

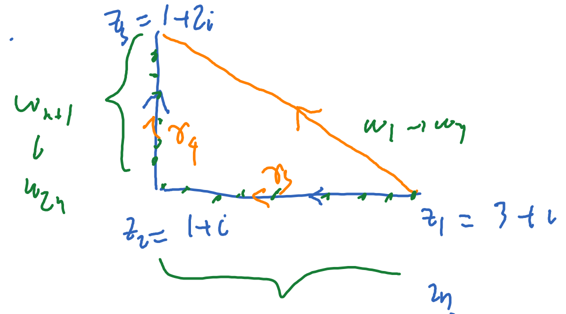
$a \le t \le b$

t

a ——— b

$$\boxed{\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt}$$

Ex.



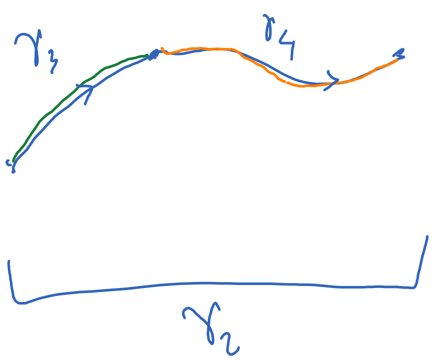
$\gamma_1 : z_1 \to z_3$

$\gamma_2 : z_1 \to z_2 \ , \ z_2 \to z_3$

$$\int_{\gamma_2} f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{2n} f(w_k)(w_{k+1}-w_k)$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n} \cdots + \sum_{k=n+1}^{2n} \cdots \right)$$

$$\int_{\gamma_2} f(z)\,dz = \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz$$



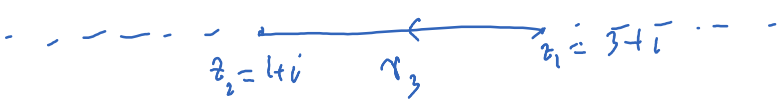
γ_2 is a continuation of γ_3 by γ_4.

" $\gamma_2 = \gamma_3 + \gamma_4$ "

$\gamma_3(t) = ?$

$z_1 \sim (\underset{x_1}{3}, \underset{y_1}{1})$

$z_2 \sim (\underset{x_2}{1}, \underset{y_2}{1})$



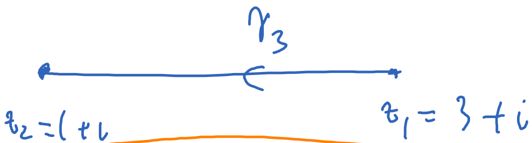
$y = mx + b$

$y = \dfrac{y_2 - y_1}{x_2 - x_1} x + b = 0x + b = b = 1$

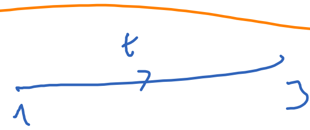
$$\begin{cases} x = t \in \mathbb{R} \\ y = 1 \end{cases}$$

parametrize eq of line

$\gamma_3(t) = x + iy = t + i.$



$$\gamma_3(t) = \cancel{t} + i \qquad 1 \le t \le 3$$



$$\gamma_3(t) = (1+3-t) + i$$

$$\boxed{\gamma_3(t) = 4 - t + i}$$

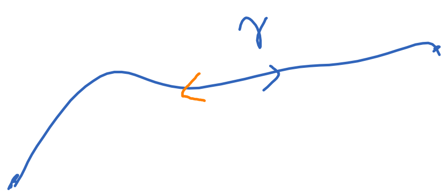
$$1 \le t \le 3$$

Several ways to switch orientation of a curve

(1) $\gamma_3(t) = -t + i \qquad -3 \le t \le -1$

(2) $\gamma(t)$ with $t \to a+b-t$ $\qquad a \le t \le b$

"$-\gamma$" opposite orientation.



$$(-\gamma)(t) = \gamma(a+b-t)$$

$$\int_{\gamma_3} f(z)\,dz = \int_1^3 f(\gamma_3(t))\,\gamma_3'(t)\,dt = \int f(\underbrace{4-t+i})(-1)\,dt$$

$$= \int_1^3 (4-t-i)(-1)\,dt$$

$$f(z) = \bar{z}$$

$$f(3-t+i) = \overline{4-t+i} = 4-t-i$$

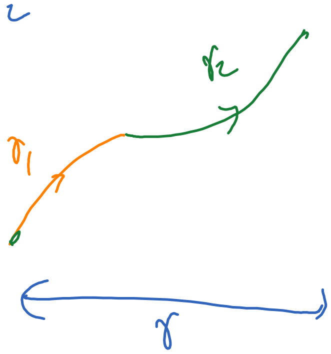
$$\int_1^3 (-4+t+i)\,dt = \underbrace{\int_1^3 (-4+t)\,dt}_{?} + i\underbrace{\int_1^3 dt}_{?}$$

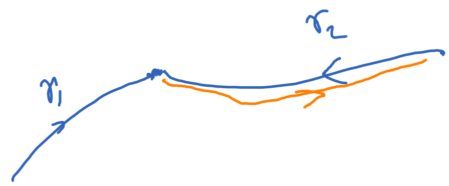
Properties of complex integral

① $$\int_a^b (u(t)+iv(t))\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

def. of int.
by Riemann sum.

② $$\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$



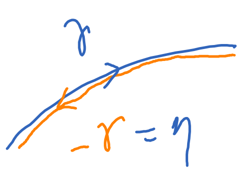


$\gamma_1 - \gamma_2$

$\gamma_1 + (-\gamma_2)$

$$\int_{\gamma_1-\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \underbrace{\int_{-\gamma_2} f(z)\,dz}_{-\int_{\gamma_2} f(z)\,dz}$$

③



$$\boxed{\int_\eta f(z)\,dz} = - \int_\gamma f(z)\,dz$$

$\eta(t) = \gamma(a+b-t)$

$\eta'(t) = -\gamma'(a+b-t)$

$$LHS = \int_a^b f(\eta(t))\,\eta'(t)\,dt$$

$$RHS = -\int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

$s = a+b-t$

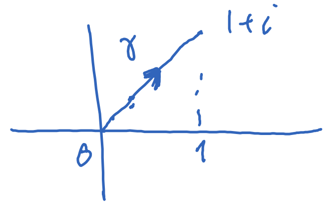
④ $$\underbrace{\int_\gamma (f(z)+g(z))\,dz}_{\lim \sum (f(z_k)+g(z_k))(z_{k+1}-z_k)} = \underbrace{\int_\gamma f(z)\,dz}_{} + \underbrace{\int_\gamma g(z)\,dz}_{}$$

⑤ $$\int_\gamma c f(z)\,dz = c\int_\gamma f(z)\,dz$$

Can we use { substitution / integration by part ?

Yes, but be careful.

Ex $$\int_\gamma \frac{z\,dz}{z^2+1}$$



$w = z^2+1$

$dw = 2z\,dz$

η is the image of γ under the map z^2+1

$$\int_\eta \frac{\frac{1}{2}dw}{w}$$

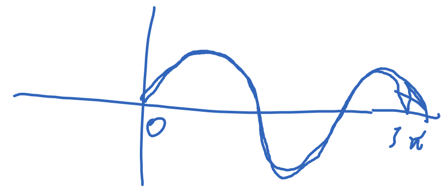
$$\int_a^b f(x)\,dx \overset{x=x(t)}{=} \int_\alpha^\beta f(x(t))\,x'(t)\,dt$$

$\int_\gamma f(z)\,dz$ — use limit def : diff

$$\int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

hard to compute

$f(z) = \frac{1}{z}$



$\gamma(t) = t + i\sin t$

Fundamental thm of calculus for complex functions.