Lecture 19

Wednesday, May 13, 2020 8:29 AM

Diff rules

$f'(z) = \lim_{z \to z_0} \frac{- -}{- -}$

f, g are differentiable at z_0.

$f+g$

fg

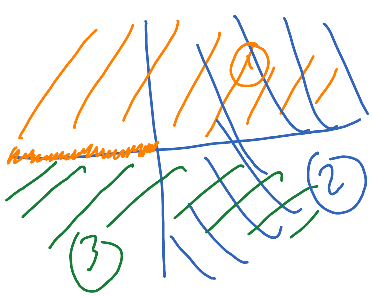
$\frac{f}{g}$ $g(z_0) \neq 0$

If f is diff at z_0 and g is diff at $f(z_0)$
then $g \circ f$ is diff at z_0.

$f(z) = \underbrace{\text{Log } z}_{u+iv} \rightsquigarrow f'(z) = \frac{1}{z}$

$u(x,y) = \ln |z| = \ln \sqrt{x^2+y^2} = \frac{1}{2} \ln(x^2+y^2)$

$v(x,y) = \text{Arg}(x+iy) = \begin{cases} \dots & (1) \\ \dots & (2) \\ \dots & (3) \end{cases}$



Thm: If $\underline{f \text{ is diff at } z_0}$ then $\underline{f \text{ is cont. at } z_0}$.

suppose

want to show $\lim_{z \to z_0} f(z) = f(z_0)$

Pick a seq. $z_n \to z_0$. Want to show $\lim_{n \to \infty} f(z_n) = f(z_0)$

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$$

$$\underbrace{\lim_{n\to\infty}(z_n-z_0)\lim_{n\to\infty}\frac{f(z_n)-f(z_0)}{z_n-z_0}} = \underbrace{f'(z_0)}\ \underbrace{\lim_{n\to\infty}(z_n-z_0)}$$

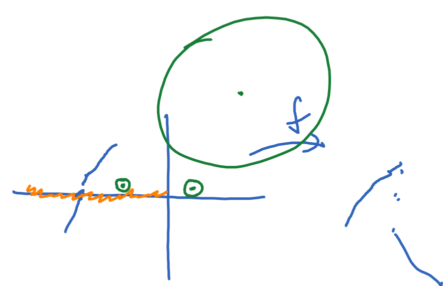
$$\lim_{n\to\infty}\left(f(z_n)-f(z_0)\right) = 0$$

$$\underbrace{f(z)-f(z_0)}_{\downarrow\ 0} = \underbrace{\frac{f(z)-f(z_0)}{z-z_0}}_{\downarrow\ f'(z_0)}\ \underbrace{(z-z_0)}_{\downarrow\ 0}$$

Ex: $f(z)=\sqrt{z}$ (principal logarithm)

where is f diff. where is f holo. $f'=?$

$$f(z) = z^{1/2} = \underbrace{e^{\frac{1}{2}\mathrm{Log}\, z}}_{\text{composite}}$$



$$z \xrightarrow{\mathrm{Log}} \underset{\substack{\uparrow\\ \text{diff}\\ \text{everywhere}}}{\xrightarrow{\text{dw. } 1/2}} \underset{\substack{\uparrow\\ \text{diff}}}{\xrightarrow{\exp}}$$

f is diff on $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$.

f is not diff on $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$ because it's not cont.

$(z^a)' = a z^{a-1}$

principal log.

$$f'(z) = \left(\frac{1}{2}\mathrm{Log}\, z\right)' e^{\frac{1}{2}\mathrm{Log}\, z} = \frac{1}{2z}\sqrt{z} = \frac{1}{2} z^{-1/2}.$$

Can we use L'Hospital rule?

$$\begin{cases} f \text{ diff at } z_0, & f(z_0)=0 \\ g \quad " \quad " \quad z_0, & g(z_0)=0 \\ g'(z_0)\neq 0 \end{cases}$$

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}. \quad ✓$$

$$\frac{f(z)}{g(z)} = \frac{f(z)-f(z_0)}{g(z)-g(z_0)} = \frac{\dfrac{f(z)-f(z_0)}{z-z_0} \to f'(z_0)}{\dfrac{g(z)-g(z_0)}{z-z_0} \to g'(z_0)}$$

In general,

$$\lim_{z\to z_0}\underbrace{\frac{f(z)}{g(z)}}_{\frac{0}{0},\ \frac{\infty}{\infty}} = \lim_{z\to z_0}\frac{f'(z)}{g'(z)}$$

Taylor expansion.

Complex integration

Antiderivative of f is a function g such that $g'=f$

anti of e^z ...... $e^z + C$

anti of $\sin z$ .. $-\cos z$

anti of $\mathrm{Log}\, z$ .... ??

$$\int \underbrace{\ln x}_{u}\, \underbrace{dx}_{v} = uv - \int v\, du$$

$u = \ln x$ $\quad du = \frac{1}{x}dx$

$v = x$ $\quad dv = dx$

$= x\ln x - \int dx$

$$\int \ln x\, dx = x\ln x - x + C$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \operatorname{Log} z \, dz$$ anti-der

$$\int_\gamma f(z)\,dz = ??$$

$u = \operatorname{Log} z$ .... $du = u'(z)\,dz$

$v = z$ $\quad dv = v'(z)\,dz$

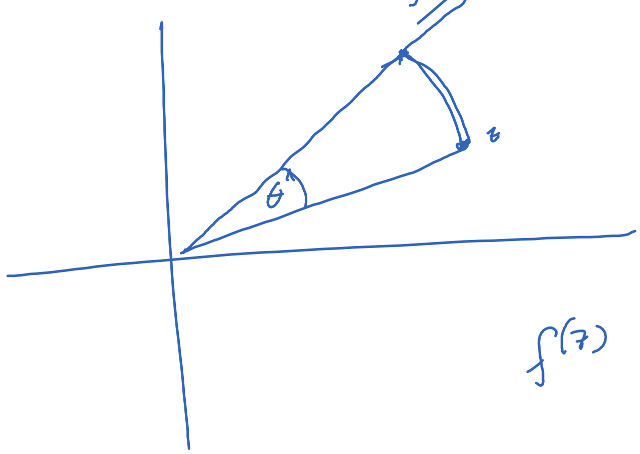
Geo. meaning of derivative.

$f : D \to \mathbb{C}$, diff at z_0.

$f(z) = az \qquad a \in \mathbb{C}$.

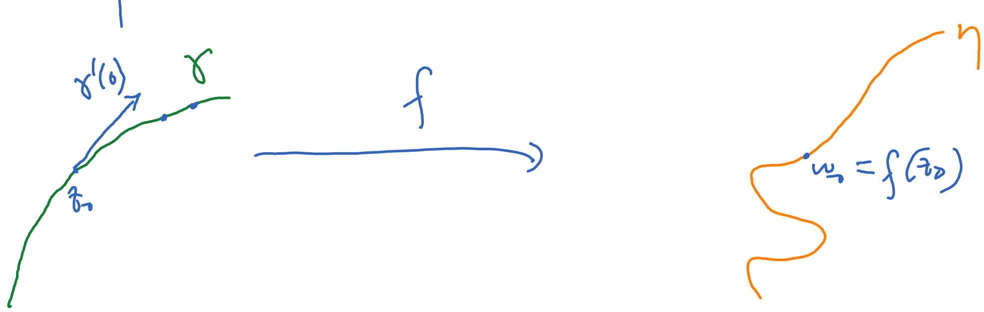
$a = re^{i\theta}$

$f(z) = \underbrace{re^{i\theta}}\, z$



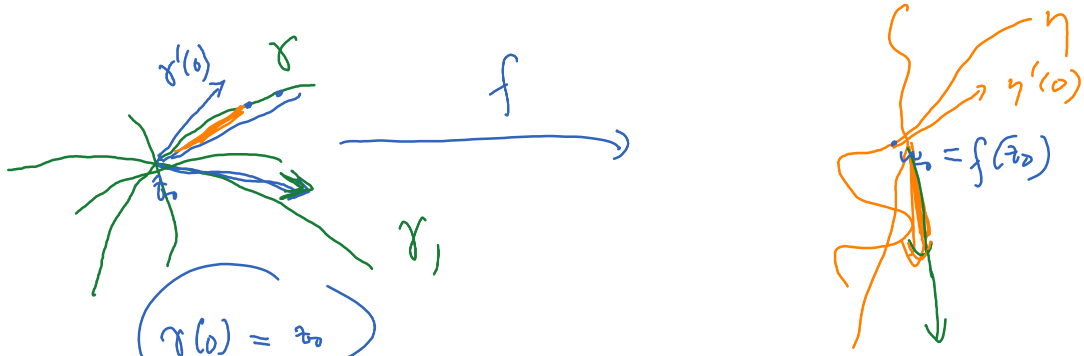
$f(z)$

$\gamma = \gamma(t)$



$\eta(t) = f(\gamma(t))$

$\gamma(0) = z_0$



$$\underbrace{\eta(t) = f(\gamma(t))}$$

$$\eta'(t) = \gamma'(t) f'(\gamma(t))$$

plug $t=0$.

$$\eta'(0) = \gamma'(0) f'(\gamma(0))$$

$$\gamma(0) = z_0$$

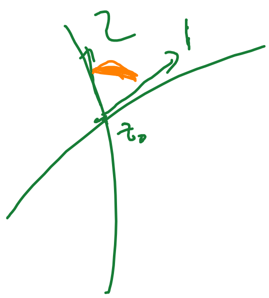
$$t_n \to 0$$

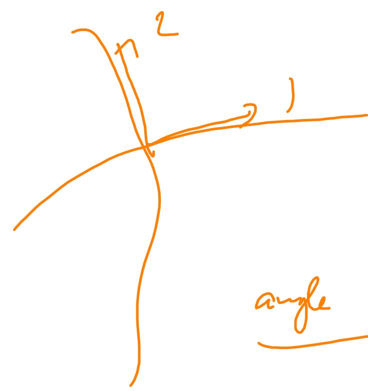
$$\frac{\gamma(t_n) - \gamma(0)}{t_n - 0} \xrightarrow{n\to\infty} \gamma'(0)$$

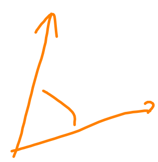
$$\eta'(0) = \gamma'(0) f'(z_0)$$

new velocity

old velocity

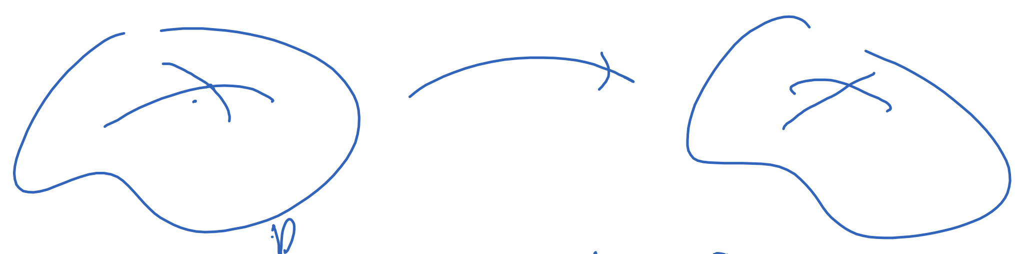






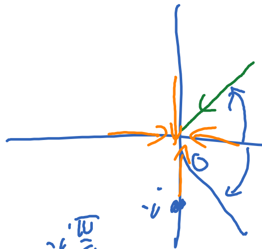
$\underbrace{\text{angle preserving}}_{\text{conformal}}$

f is said to be <u>conformal</u> in D if f is angle preserving



<u>Thm</u>

f is conformal in D iff $\begin{cases} f \text{ is diff in } D \\ f'(z) \neq 0 \quad \forall z \in D. \end{cases}$



$z = re^{i\frac{\pi}{4}}$

$$f(z) = \frac{\bar{z}}{z}\, \underbrace{e^{i\,\mathrm{Arg}(z^2)}}_{\to\, e^{i\frac{\pi}{2}} = i}$$

$\bar{z} = re^{-i\frac{\pi}{4}}$

$$\frac{\bar{z}}{z} = \frac{re^{-i\pi/4}}{re^{i\pi/4}} = e^{-i\frac{\pi}{2}} = -i$$

$z^2 = r^2 e^{i\frac{\pi}{2}}$

$z = ia$

$z^2 = -a^2$

$\mathrm{Arg}(z^2) = \pi$

If $z \in i\mathbb{R}$ then $\mathrm{Arg}(z^2) = \pi$ and $e^{i\,\mathrm{Arg}(z^2)} = e^{i\pi} = -1$

If $z \in \mathbb{R}$ then $\mathrm{Arg}(z^2) = 0$ and $e^{i\,\mathrm{Arg}(z^2)} = e^0 = 1$.

$$f(z) = 1 \text{ on } \begin{cases} \mathbb{R} \\ i\mathbb{R} \end{cases}$$

$$\frac{z}{|z|}$$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$f(z) = u(x,y) + i\,v(x,y)$$

$$f'(z_0) = \underline{u_x(x_0, y_0) + i v_x(x_0, y_0)} = \underline{v_y - i u_y}$$

$z_0 = x_0 + iy_0$