Lecture 18

Monday, May 11, 2020 9:01 AM

$$f'(z_0) = \lim_{z \to z_0} \underbrace{\frac{f(z) - f(z_0)}{z - z_0}}$$

$$\frac{0}{0}$$

$$f(z) = \frac{1}{z}$$

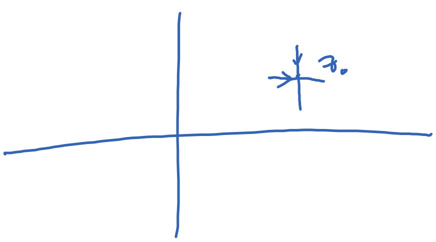
$$\underbrace{\frac{\frac{1}{z} - \frac{1}{z_0}}{z - z_0} = -\frac{1}{z z_0}} \xrightarrow{z \to z_0} -\frac{1}{z_0^2}$$

$$f(z) = e^z$$

$$f(z) = \sin z$$

$$\ldots$$

$$f(z) = \frac{z+1}{z+2}$$



$$f'(z_0) = \lim_{\substack{z \to z_0 \\ z - z_0 \in \mathbb{R}}} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{\substack{t \to 0 \\ t \in \mathbb{R}}} \frac{f(z_0 + t) - f(z_0)}{t}$$

$$= u_x(z_0) + i v_x(z_0)$$

$$f(z) = u(z) + i v(z).$$

$$f'(z_0) = u_x(z_0) + i v_x(z_0)$$

$$= v_y(z_0) - i u_y(z_0)$$

C-R theorem:

$$f \text{ is diff at } z_0 \text{ iff } \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \text{ at } z_0.$$

Ex

$$f(z) = \bar{z}$$

$$z = x + iy$$

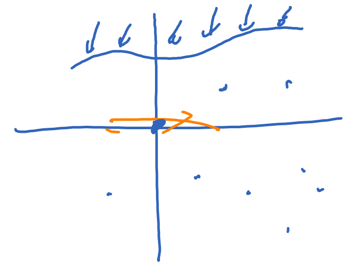
$$f(z) = \underbrace{x}_{u(x,y)} - iy \qquad v(x,y) = -y$$

Ex:

$$f(z) = \bar{z}^2$$

$$z = x + iy$$

$$f(z) = (x - iy)^2 = x^2 - y^2 - i\,2xy$$



$$u(x,y) = x^2 - y^2$$

$$f(z) = \underbrace{u(x,y)} + i\,\underbrace{v(x,y)}$$

$$v(x,y) = -2xy$$

f is diff at $z = x + iy$ iff

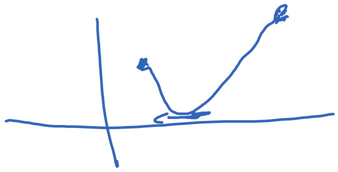
$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \rightsquigarrow \begin{cases} 2x = -2x \\ -2y = 2y \end{cases} \rightsquigarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$\underset{z=0}{f'(0)} = u_x(\underset{z=0}{0}) + i\,v_x(0) = 0 + i0 = 0.$$

$u_x = 2x$ ✓

$v_x = -2y$ ✓

Cal I: $f : (a,b) \to \mathbb{R}$ diff.



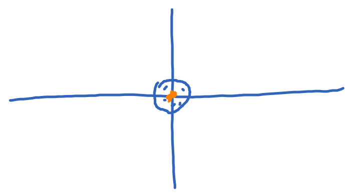
$$\boxed{f'(x) = 0}$$

Differentiable ⟶ $f'(z_0) = \lim_{z \to z_0} \sim$ exists

Def

f is holomorphic at z_0 if f is diff. on a disk centered at z_0

$f(z) = \bar{z}^2$ is nowhere holomorphic.



Ex:

$$f(z) = e^z$$

$$z = x + iy$$

$$f(z) = e^{x+iy} = e^x(\cos y + i \sin y) = \underbrace{e^x \cos y}_{u(x,y)} + i\, \underbrace{e^x \sin y}_{v(x,y)}$$

$$\begin{cases} u_x = e^x \cos y = v_y = e^x \cos y \\ u_y = -e^x \sin y = -v_x = -e^x \sin y \end{cases}$$

f is entire if f is diff. at all $z \in \mathbb{C}$.

$$f'(z) = u_x + i v_x = u + iv = f(z) = e^z$$

$$\frac{d}{dz} e^z = e^z$$

Check if f is diff at z_0

- $$\lim \frac{f(z)-f(z_0)}{z-z_0}$$

- $$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \qquad \begin{aligned} f'(z) &= u_x + i v_x \\ &= v_y - i u_y \end{aligned}$$

Differential rules :

$$\frac{d}{dz}(f+g) = \frac{df}{dz} + \frac{dg}{dz} \qquad (f+g)' = f' + g'$$

$$(fg)' = f'g + fg' \qquad c' = 0$$

$$\underbrace{(f \circ g)'(z_0)} = f'(g(z_0))\, g'(z_0) \qquad (cf)' = cf'$$

$$\lim \underbrace{\frac{f(g(z)) - f(g(z_0))}{z - z_0}}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$\underbrace{\frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)}} \quad \underbrace{\frac{g(z) - g(z_0)}{z - z_0}}$$

$$\to f'(g(z_0)) \qquad \to g'(z_0)$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\frac{d}{dz} e^{iz} = (iz)' e^{iz} = i e^{iz}$$

$$\frac{d}{dz}\sin z = \frac{i e^{iz} - (-i) e^{-iz}}{2i}$$

$$z \xrightarrow{\times i} iz \xrightarrow{\exp} e^{iz}$$

$$= \frac{e^{iz} + e^{-iz}}{2}$$

$$= \cos z .$$

$$e^{\operatorname{Log} z} = z$$

$$g(z) = e^z$$

$$f(z) = \operatorname{Log} z$$

$$g(f(z)) = z$$

$$g'(f(z))\, \underline{f'(z)} = 1$$

$$(z^n)' = n z^{n-1}$$

$$(z^2)' = (zz)' = \underbrace{z'}_{1} z + z \underbrace{z'}_{1} = 2z$$