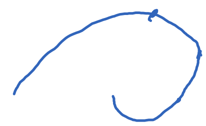
Lecture 17

Friday, May 8, 2020 9:00 AM

$f:(a,b)\to\mathbb{R}$ $\qquad f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$f:(a,b)\to\mathbb{R}^2$ $\qquad f(t)=(u(t),v(t))$



$f'(t)=(u'(t),v'(t))$

$f:\mathbb{R}^2\to\mathbb{R}^2$ $\qquad f(x,y)=(u(x,y),v(x,y))$

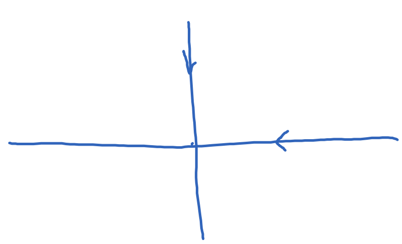
$$\underbrace{\underbrace{J_f(x,y)}_{D_f(x,y)}}_{\text{derivative matrix}}=\begin{bmatrix}u_x & u_y\\ v_x & v_y\end{bmatrix}$$

$u_x=\frac{\partial u}{\partial x}$

$f:\mathbb{C}\to\mathbb{C}$ $\qquad f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$

$$=\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$$

It is quite demanding / hard for a function to be differentiable at a point.



$f(z)=\underbrace{u(z)}+i\underbrace{v(z)}$

$\sim\ u(x,y)\qquad v(x,y)$

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$$

$h=t\in\mathbb{R}$

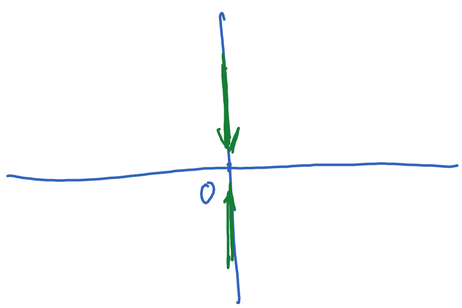
$$\frac{f(z+t)-f(z)}{t} = \frac{f(x+t+iy)-f(x+iy)}{t}$$

$$= \frac{u(x+t,y)+iv(x+t,y)-[u(x,y)+iv(x,y)]}{t}$$

$$= \underbrace{\frac{u(x+t,y)-u(x,y)}{t}}_{\downarrow\, t\to 0} + i\,\underbrace{\frac{v(x+t,y)-v(x,y)}{t}}_{\downarrow}$$

$$\frac{\partial u}{\partial x}(x,y) = u_x \qquad\qquad v_x$$

$$\lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(z+h)-f(z)}{h} = u_x + iv_x$$



$$h = it$$

$$\lim_{t\to 0} \frac{f(z+it)-f(z)}{it}$$

$$\frac{f(z+it)-f(z)}{it} = \frac{f(x+i(y+t))-f(x+iy)}{it}$$

$$= \frac{u(x,y+t)+iv(x,y+t)-[u(x,y)+iv(x,y)]}{it}$$

$$\frac{1}{i} = -i$$

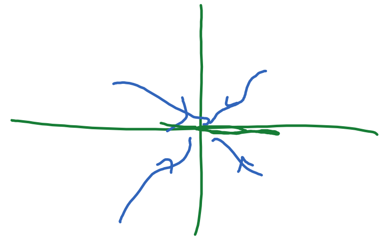
$$= -i\,\frac{u(x,y+t)-u(x,y)}{t} + \frac{v(x,y+t)-v(x,y)}{t}$$

$$\longrightarrow -iu_y + v_y$$

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z+h) - f(z)}{h} = u_x + i v_x$$

$$\Updownarrow$$

$$\lim_{\substack{h \to 0 \\ h \in i\mathbb{R}}} \frac{f(z+h) - f(z)}{h} = v_y - i u_y$$



Thm:

If $\underbrace{f \text{ is diff. at } z}_{\lim_{h \to 0} \frac{f(z+h)-f(z)}{h} \text{ exists}}$ then $\underbrace{\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}}_{\text{Cauchy-Riemann equations}}$ at point $z = x + iy$

Ex:

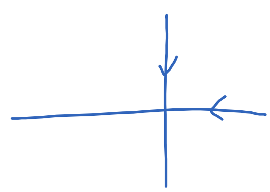
$f(z) = \bar{z}$

Find z's where f is differentiable. Find $f'(z)$

(1) use def.

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{\overline{z+h} - \bar{z}}{h} = \lim_{h \to 0} \frac{\bar{h}}{h}$$

$$\lim_{h \to 0} \frac{\bar{h}}{h} \quad \text{DNE}$$



(2) $f(z) = \bar{z} = \overline{x+iy} = x - iy$

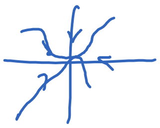
$f(x,y)$

$z = x + iy$

$$\left.\begin{aligned} u(x,y) &= x \\ v(x,y) &= -y \end{aligned}\right\}$$

$u_x = 1$; $v_y = -1$

$u_y =$; $v_x =$



If $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ at point z then $\underbrace{f \text{ is diff. at } z}_{\lim\limits_{h\to 0} \frac{f(z+h)-\dots}{\dots} \text{ exists}}$.

$$f'(z) = u_x + i v_x \text{ ``}= \frac{\partial f}{\partial x}\text{''}$$

$$= v_y - i u_y \text{ ``}= -i\frac{\partial f}{\partial y}\text{''}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(u + iv) = u_y + i v_y$$

$$-i\frac{\partial f}{\partial y} = -i u_y + v_y$$

Vector calculus:

If $f: \mathbb{R}^2 \to \mathbb{R}$ has $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ at (x_0, y_0) then

$$\frac{d}{dt}\left[f(\gamma(t))\right] \text{ exists} \quad (\text{provided that } \gamma' \text{ exist})$$

$$\| $$

$$\underbrace{D_f(\gamma(t))}_{\text{matrix}} \underbrace{\gamma'(t)}_{\text{vector}}$$