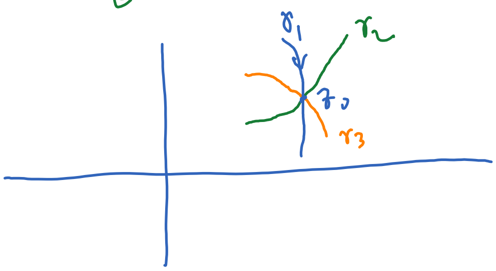
Lecture 16

Wednesday, May 6, 2020 8:59 AM

$(*)$ $\left[\lim_{z\to z_0} f(z) = L\right]$ iff $\left[\text{for any sequence } z_n \to z_0, \text{ we have } f(z_n) \to L.\right]$



$\gamma_1(t) \qquad t \in \mathbb{R}$

$f(\underbrace{\gamma_1(t)}_{z})$ $\qquad z_0 = \gamma_1(t_0)$

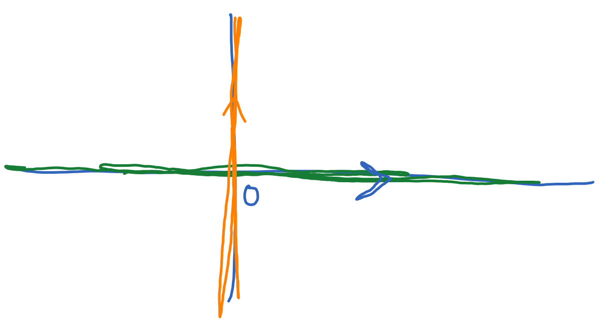
$$\lim_{t\to t_0} f(\gamma_1(t)) = L$$

Equivalent def. of $(*)$:

$\Big[$ For any path γ that passes through z_0

$\gamma(t_0) = z_0$,

we have $\displaystyle\lim_{t\to t_0} f(\gamma(t)) = L.$ $\Big]$



Ex: $f(z) = \dfrac{\bar z}{z}$

$\displaystyle\lim_{z\to 0} f(z) = ?$

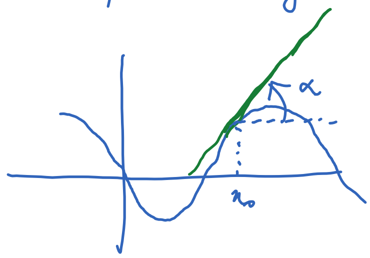
$\gamma_1(t) = t + 0i = t$

$f(\gamma_1(t)) = f(t) = \dfrac{\bar t}{t} = \dfrac{t}{t} = 1$ $\times$

$\gamma_2(t) = it = 0 + it$

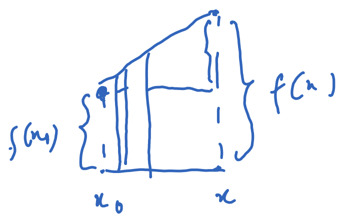
$f(\gamma_2(t)) = f(it) = \dfrac{\overline{it}}{it} = \dfrac{-it}{it} = -1$ $\times$

Calculus $\left\{ \begin{array}{l} \text{- limit} \\ \text{- continuity of functions} \\ \text{- derivative} \;\ldots\ldots \\ \text{- integral} \end{array} \right.$



$f(x) = \sin x$

$$f'(x_0) = \tan\alpha = \underbrace{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}_{\text{instant rate of change}}$$



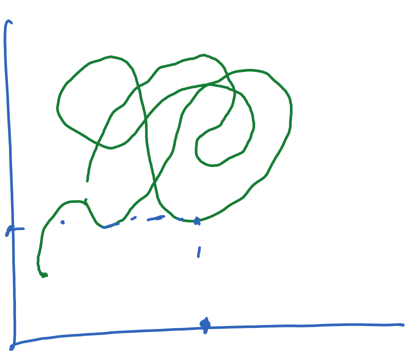
Derivative of function $f(z)$ at z_0 is defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$ ✓

$f : [a, b] \to \mathbb{R}$

$f : [a, b] \to \mathbb{R}^2 \quad \ldots\ldots \quad f(t) = (u(t), v(t))$

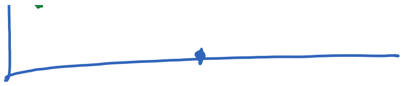
$$f'(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} = \lim_{t \to t_0} \frac{(u(t), v(t)) - (u(t_0), v(t_0))}{t - t_0}$$



$$= \lim_{t \to t_0} \frac{(u(t) - u(t_0),\; v(t) - v(t_0))}{t - t_0}$$

$$= \lim_{t \to t_0} \left(\frac{u(t) - u(t_0)}{t - t_0},\; \frac{v(t) - v(t_0)}{t - t_0} \right)$$

$$= \left(\lim \frac{u(t) - u(t_0)}{t - t_0},\; \lim \frac{v(t) - v(t_0)}{t - t_0} \right)$$



$t \to t_0$

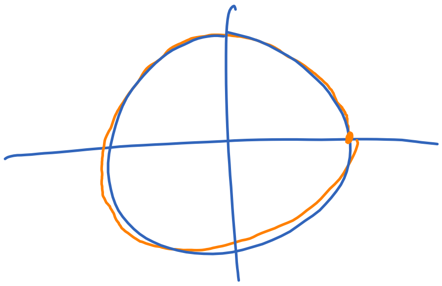
$$= \left(\lim \frac{u(t) - u(t_0)}{t - t_0},\ \lim \frac{v(t) - v(t_0)}{t - t_0} \right)$$

$$= (u'(t_0),\ v'(t_0))$$

$f : [a,b] \to \mathbb{R}^2$ - - - - -> a parametrized path / curve

$f(t) = (u(t), v(t))$

Then $f'(t) = (u'(t), v'(t))$



$(\cos(t), \sin(t))$ $t \in [0, 2\pi]$

$(\cos(2t), \sin(2t))$ $t \in [0, \pi]$.

$(\cos(t^2), \sin(t^2))$ $t \in [0, \sqrt{2\pi}]$

$\gamma(t) = x(t) + i\,y(t)$ .... complex parametrization of | curve / path

$\gamma_1(t) = \underbrace{\cos t + i \sin t} = e^{it}$

$\gamma_2(t) = e^{i2t}$

$\gamma(t) = \gamma_1(t) + \gamma_2(t) = \underbrace{e^{it} + e^{i2t}} = x(t) + i\,y(t)$

$\cos t + i \sin t$

$\cos 2t + i \sin 2t$

$\{x(t), y(t)\}$

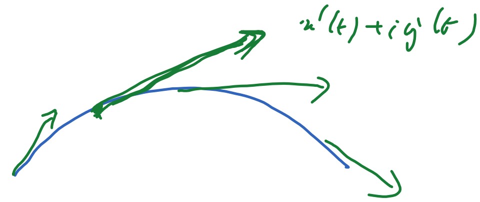
$= \mathrm{Re}\,\mathrm{Im}[\gamma(t)]$

$$\gamma'(t_0) = \lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} = \lim_{t \to t_0} \frac{x(t) - x(t_0) + i(y(t) - y(t_0))}{t - t_0}$$

$$= \lim \frac{x(t) - x(t_0)}{t - t_0} + i \lim \frac{y(t) - y(t_0)}{t - t_0}$$

$$= x'(t) + i\,y'(t)$$

$\gamma(t) = x(t) + iy(t)$



$\gamma'(t) = x'(t) + iy'(t)$

tangent vector to γ

velocity of γ.

$f : [a,b] \to \mathbb{R}$

$$\left\{ \begin{array}{l} f : [a,b] \to \mathbb{R}^2 \\ f(t) = (u(t), v(t)) \end{array} \right. \quad \cdots\cdots \quad \left. \begin{array}{l} \gamma : [a,b] \to \mathbb{C} \\ \gamma(t) = u(t) + iv(t) \end{array} \right\} \text{param path}$$

$f : \mathbb{C} \to \mathbb{C}$, $\quad f(z) = e^z$

$f'(z) = ?$

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

e^z $\quad \sqrt{z}$ $\quad \tan \sqrt{z}$

$\sin z$