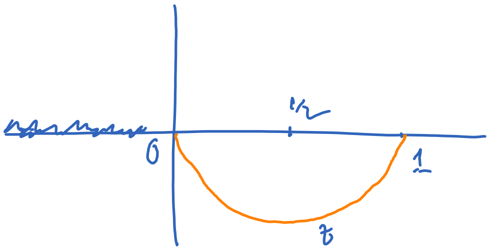
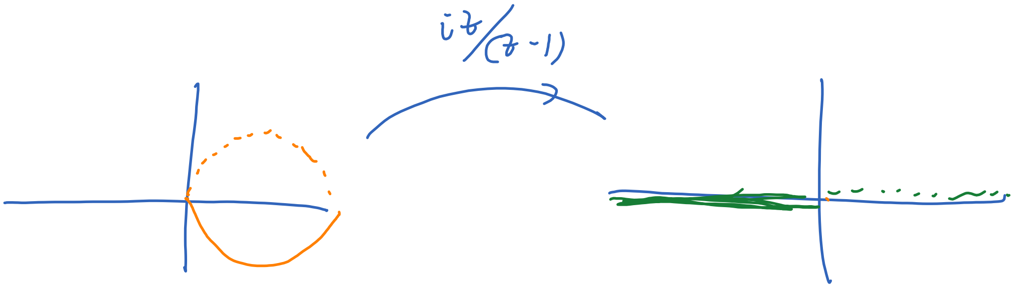
Lecture 15

Friday, May 1, 2020 8:59 AM

$$f(z) = \left(\frac{iz}{z-1}\right)^z = e^{z \operatorname{Log}\frac{iz}{z-1}}$$

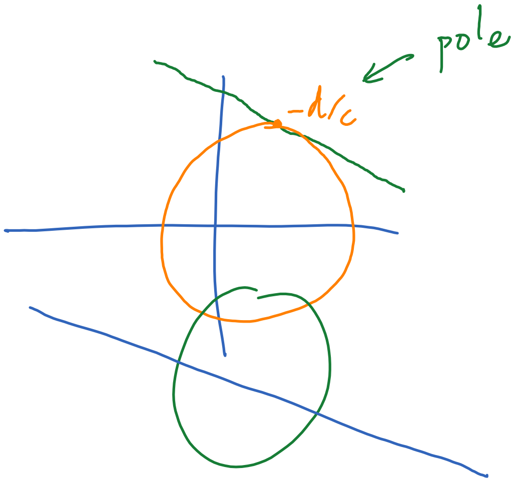


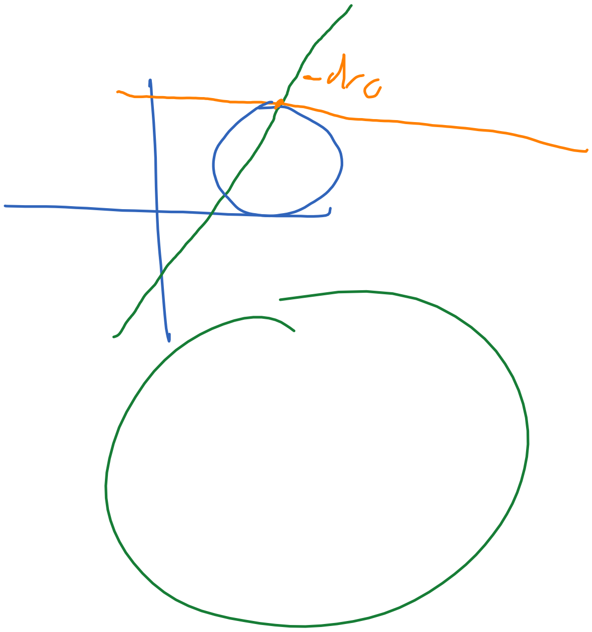
$$\underbrace{\frac{iz}{z-1} \in \mathbb{R}_{\leq 0}}_{z \in \frac{t^2}{t^2+1} + i\frac{t}{t^2+1}} \qquad (t \leq 0)$$



$$\frac{iz}{z-1}$$

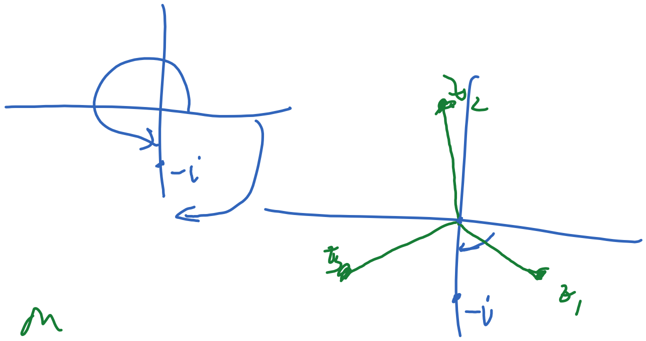
$$\boxed{f(z) = \frac{az+b}{cz+d} \quad \text{linear fractional function.}}$$





$$\frac{z+1}{z^3+i}$$

$$z^3 = -i = e^{-i\frac{\pi}{2}} = e^{i\frac{3\pi}{2}}$$



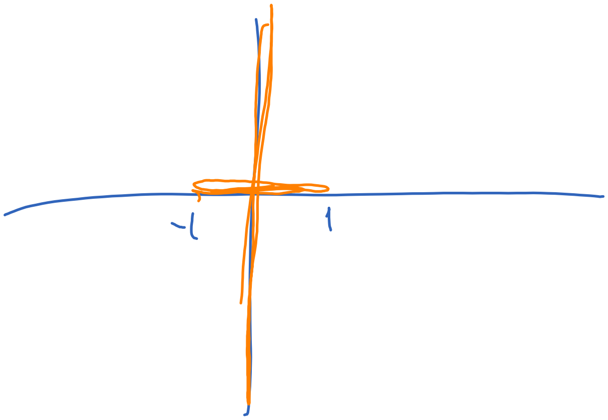
Cont on

$\mathbb{C} \setminus \{z_1, z_2, z_3\}$

$$\sqrt{z^2-1}$$

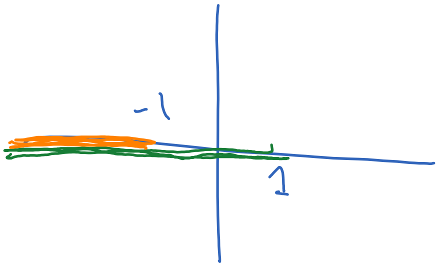
$$\parallel$$

$$e^{\frac{1}{2}\log(z^2-1)}$$



$z^2-1 \in \mathbb{R}_{\le 0}$

$$\underbrace{\sqrt{z+1}}\ \underbrace{\sqrt{z-1}}$$



$z+1 \in \mathbb{R}_{\le 0}$

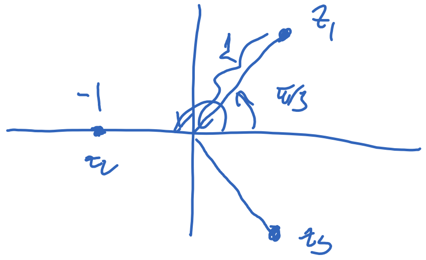
$z+1 = t \le 0$

$z = t-1 \le -1$

$$\frac{z}{z^3+1}$$

$$z^3+1=0 \rightsquigarrow z^3=-1=e^{i\pi}$$

$$z=\sqrt[3]{-1}=$$



$$z_1 = e^{i\frac{\pi}{3}}$$

$$z_2 = e^{i\pi} = -1$$

$$z_3 = e^{i\frac{4\pi}{3}}$$

$$\underbrace{z^3+1}=(z-z_1)(z-z_2)(z-z_3)$$

$$\frac{z}{z^3+1}=\frac{z}{(z-z_1)(z-z_2)(z-z_3)}$$

$$\boxed{\lim_{z\to i}\frac{z}{z^2+1}}$$

$$\begin{tikzcd} & e^{i\pi/3} \\ \frac{z}{z^3+1} \arrow[ur] \arrow[dr] & \\ & 0 \end{tikzcd}$$

$$\lim_{x\to 0}\frac{1}{x} \quad \text{DNE}$$

$$\lim_{x\to 0}\frac{1}{x^2}=\infty$$

$$\lim_{x\to 0}\frac{1}{-x^2}=-\infty$$

We want to show that

$$\lim_{z\to e^{i\frac{\pi}{3}}}\boxed{\frac{z}{z^3+1}}=\infty$$

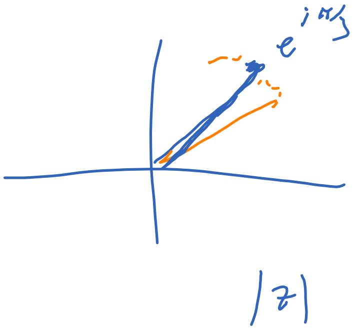
Let z_n be a seq. that goes to $e^{i\frac{\pi}{3}}$

want to show: $\dfrac{z_n}{z_n^3+1}\to\infty$

want to show $\left|\dfrac{z_n}{z_n^3+1}\right|\to\infty$.

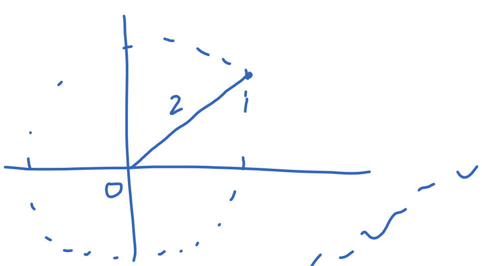
$$\left|\frac{z_n}{z_n^3+1}\right| = \frac{|z_n| \to 1}{|z_n^3+1| \to 0^+} \longrightarrow \infty$$

$z_n \to e^{i\pi/3}$, so $|z_n| \longrightarrow |e^{i\pi/3}| = 1$



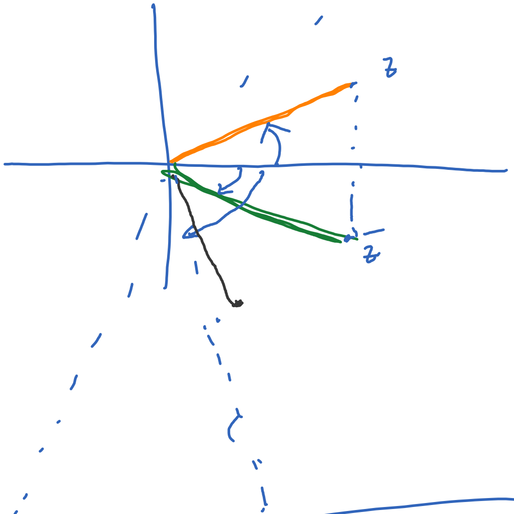
$z_n^3+1 \longrightarrow (e^{i\pi/3})^3+1 = 0$

$$\frac{a_n \to 1}{b_n \to 0^+} \longrightarrow \infty$$



$$\left|\frac{\bar z}{z}\right|$$

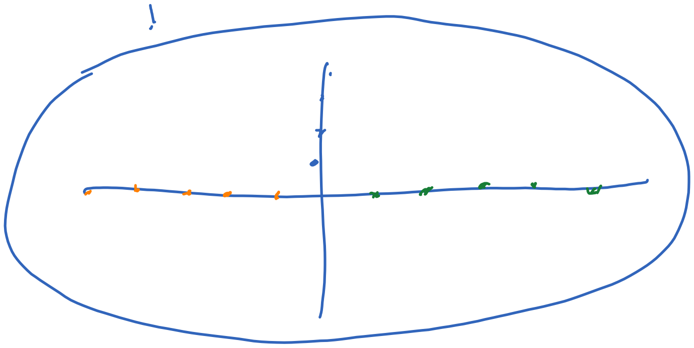
$$= \frac{|\bar z|}{|z|} = 1$$



$$\frac{\bar z}{z}$$

$z \to \infty$

$|z| \to \infty$



$z_n = n$

$$f(z_n) = \frac{\bar z_n}{z_n} = \frac{\bar n}{n} = 1$$

$w_n = -n$

$$f(w_n) = \frac{\bar w_n}{w_n} = \frac{\overline{-n}}{-n} = 1$$

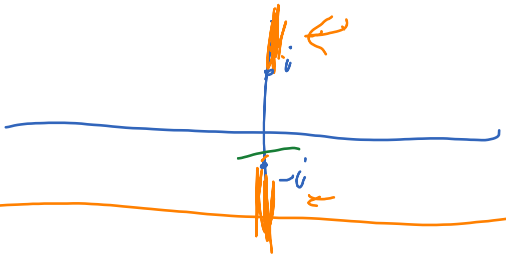
$u_n = in$

$$f(u_n) = \frac{\overline{in}}{in} = \frac{-in}{in} = -1$$

$\operatorname{Log}(z^2+1)$

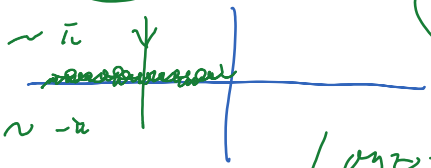
$z^2+1 \in \mathbb{R}_{\leq 0}$.

$$\begin{cases} y \leq -1 \text{ or } y \geq 1 \\ x = 0 \end{cases}$$



$$\boxed{(z^2+1)^i = \underbrace{e^{i \operatorname{Log}(z^2+1)}}}$$

$$z^2 = e^{2 \operatorname{Log} z}$$



Log z jumps by 2πi

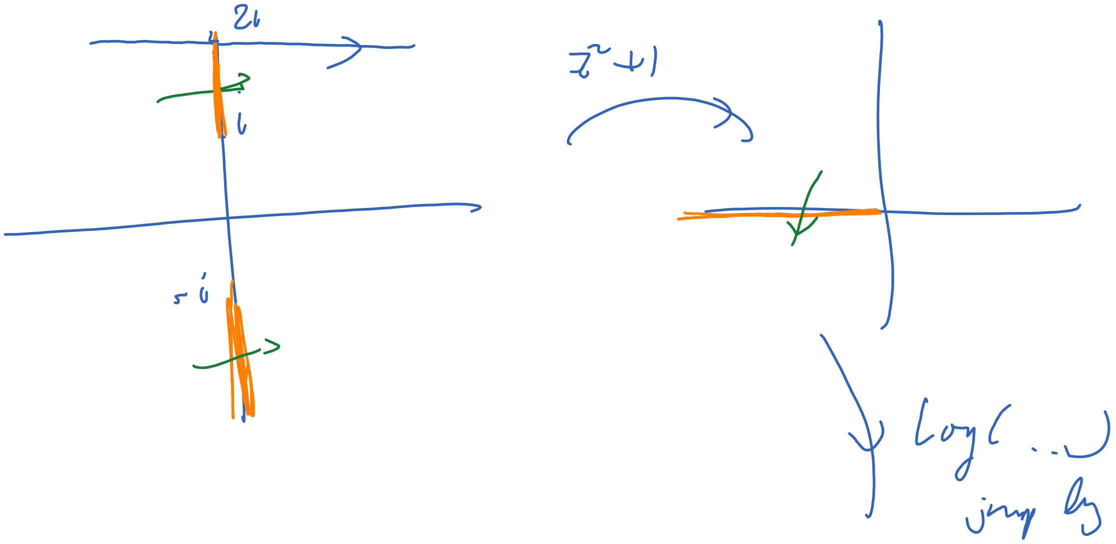
$\operatorname{Log} z = \ln|z| + i \underline{\operatorname{Arg} z}$.

$\sim 2\pi$

2 Log z jumps by 4πi

$e^{2\operatorname{Log} z}$ scaled by $e^{4\pi i} = 1$.

$$e^{u+h} = e^u \underline{e^h}$$



$\text{Log}(\ldots)$

jump by $2\pi i$

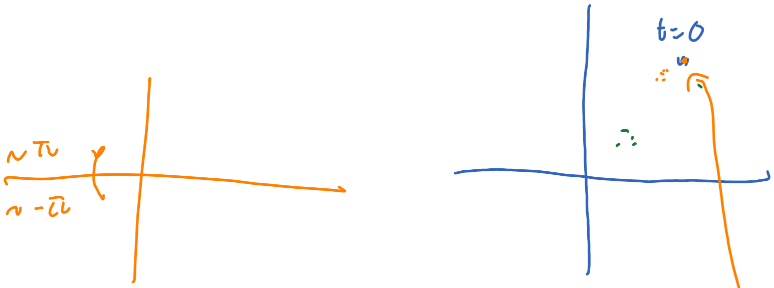
$i\text{Log}(\ldots)$

jump by $i\,2\pi i$

$= -2\pi$

$e^{i\text{Log}(\ldots)}$

scaled by $\underbrace{e^{-2\pi}}_{\neq 1}$



?[S]

P[S]

$z \sim 2i$

-1 5

$x = t$

$y = 2$

$\boxed{z = t + 2i}$

$(z^2+1)^i = e^{i\text{Log}(z^2+1)} = e^{i\text{Log}((t+2i)^2+1)}$

Plot[ReIm[E^

]

```
p[s_]:=ParametricPlot[ReIm[En                ],
                              [t+2I]   {t, -1, s}]
q[s_]:=
        Manipulate[p[s], {s,-0.5, 0.5}]
                  q[s]
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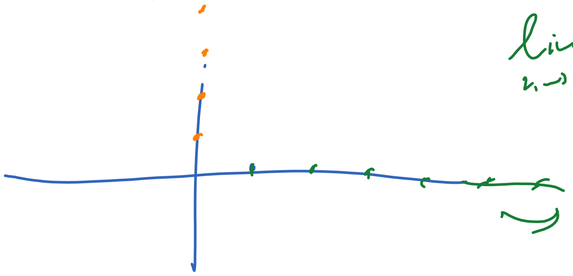
$$\lim_{z \to \infty} \frac{\sin z}{z}$$



$$\lim_{n \to \infty} \frac{\sin n}{n} = 0$$

$$\lim_{x \to \infty} \frac{\sin x}{x} = 1$$