Lecture 14

Wednesday, April 29, 2020 8:56 AM

$f(z) = \text{Log}\, z = \ln|z| + i\,\text{Arg}\, z$

f is cont at z_0 iff $\begin{cases} \ln|z| \text{ is cont at } z_0 \\ \text{Arg}\, z \text{ is cont at } z_0 \end{cases}$

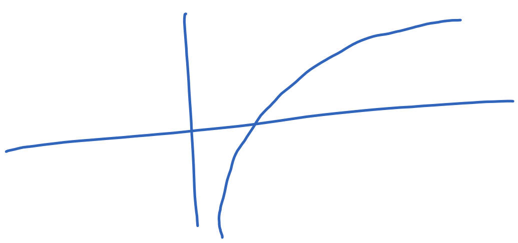
$|z|$ .... is cont everywhere.

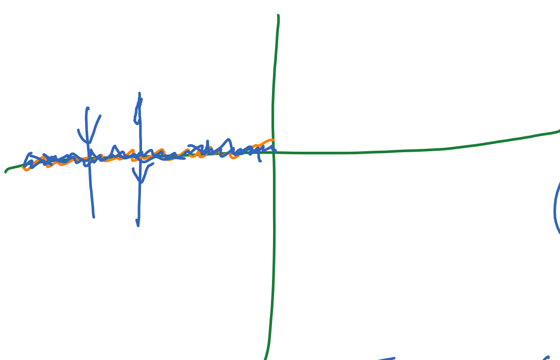
$|z| = \underbrace{\sqrt{x^2+y^2}}_{u(x,y)} + i\underbrace{0}_{v(x,y)}$

$\ln$ cont on $(0,\infty)$

$\cancel{\ln(-1)}$ $\text{Log}(-1)$

$\ln|z|$ is cont. on $\underline{\underline{\mathbb{C}\setminus\{0\}}}$.





Arg z is cont. on $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$.

$f(z) = \text{Log}\, z$ is cont on $\mathbb{C}\setminus\mathbb{R}_{\leq 0}$.

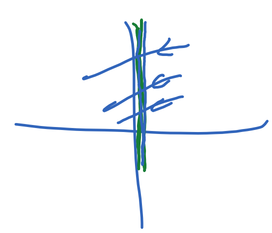
(principal logarithm).

$f(z) = \sqrt{iz-1}$

What is the region of continuity of f?

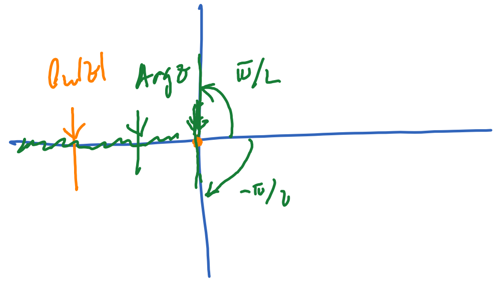
$\sqrt{iz-1} = (iz-1)^{1/2} = e^{\frac{1}{2}\text{Log}(iz-1)}$.

e^z is cont everywhere



e^z is cont everywhere on $\mathbb{C}$

$$\operatorname{Log} z = \underbrace{\ln|z|}\, + i\,\underbrace{\operatorname{Arg} z}.$$



$\operatorname{Log} z$ jumps $2\pi i$ across the line $\mathbb{R}_{\leq 0}$

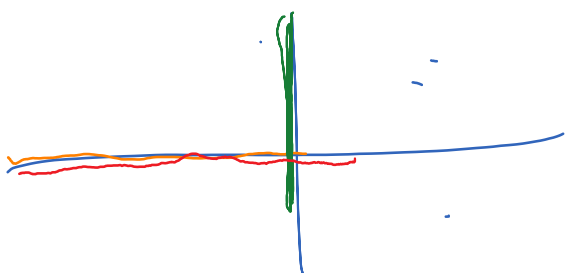
$\operatorname{Log} z$ jumps by πi " " origin along the $\{x=0\}$.

$\ln|z|$ varies continuously

$$(iz-1)^{1/2} = e^{\frac{1}{2}\operatorname{Log}(iz-1)}$$

$$\underbrace{z \xrightarrow{iz-1} iz-1}_{\text{cont}} \;\underbrace{\xrightarrow{\operatorname{Log}}}\; \underbrace{\operatorname{Log}(iz-1) \xrightarrow{\times\frac{1}{2}} \frac{1}{2}\operatorname{Log}(iz-1)}_{\text{cont}}$$

$$\left.\frac{1}{2}\operatorname{Log}(iz-1) \xrightarrow{\exp} e^{\frac{1}{2}\operatorname{Log}(iz-1)}\right\}\text{cont}$$



$i = e^{i\frac{\pi}{2}}$

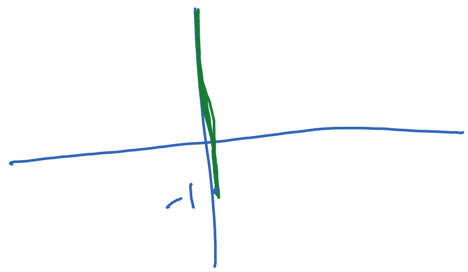
$zi = z e^{i\frac{\pi}{2}}$

potential points of discontinuity: z such that $iz-1 \in \mathbb{R}_{\leq 0}$

$z = x+iy$

$$iz-1 = \underbrace{-y-1}_{\leq 0} + i\underbrace{x}_{=0} \in \mathbb{R}_{\leq 0}$$

$$\begin{cases} y \geq -1 \\ x = 0 \end{cases}$$



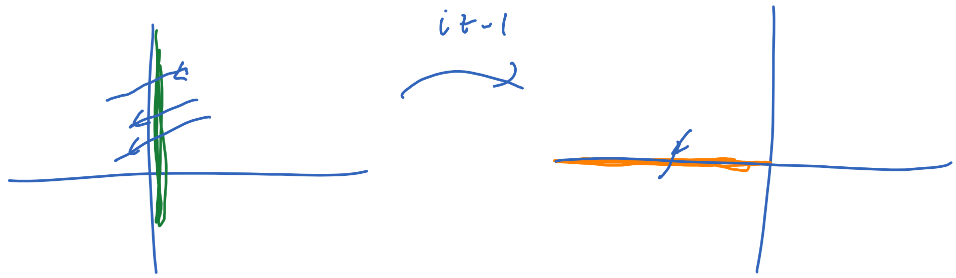
$iz - 1 \in \mathbb{R}_{\leq 0}$

$iz - 1 = t \leq 0$

$$z = \frac{t+1}{i} = -i(t+1) = i(\underbrace{-t-1}_{s}) = is$$

$$t \leq 0 \longrightarrow s \geq -1$$

$iz - 1 \in \mathbb{R}_{\leq 0} \rightsquigarrow z = is$ where $s \geq -1$.



$\mathrm{Log}(iz-1)$ jumps by $2\pi i$

$\Big\downarrow \frac{1}{2}$

$\frac{1}{2}\mathrm{Log}(iz-1)$ jump by πi

$\Big\downarrow \exp$

$e^{\frac{1}{2}\mathrm{Log}(iz-1)}$ scaled by $e^{\pi i} = -1$

$$e^{u+v} = e^u e^v$$

Conclusion: $\sqrt{iz-1}$ is cont everywhere except on the line

$$\underbrace{\begin{cases} x = 0 \\ y \geq -1 \end{cases}}_{}$$

$$z = iy \quad (y \geq -1)$$

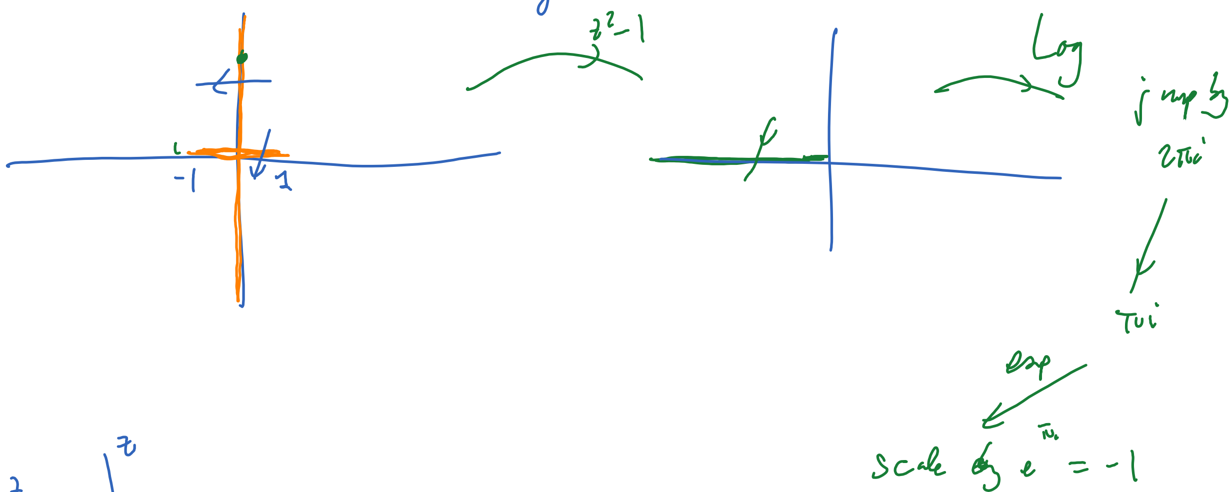
$$z^z = e^{z \,\mathrm{Log}\, z}$$

$$z \xrightarrow{\text{cont}} z\,\mathrm{Log}\, z \xrightarrow{\text{cont}} e^{z\,\mathrm{Log}\, z} \quad \text{Cont} \quad \mathbb{C}$$

$$\sqrt{z^2-1} = e^{\frac{1}{2}\mathrm{Log}\,(z^2-1)}$$

find z such that $z^2-1 \in \mathbb{R}_{\le 0}$.

$z = x + iy$



$$f(z) = \left(\frac{iz}{z-1}\right)^z$$

$$= e^{z\,\mathrm{Log}\left(\frac{iz}{z-1}\right)}$$

$$\frac{iz}{z-1} \in \mathbb{R}_{\le 0}.$$

$$\frac{iz}{z-1} = t \le 0$$

$$iz = zt - t$$

$$z = \frac{t}{t-i} = \frac{t(t+i)}{t^2+1} = \underbrace{\frac{t^2}{t^2+1}}_{x(t)} + i\underbrace{\frac{t}{t^2+1}}_{y(t)}$$