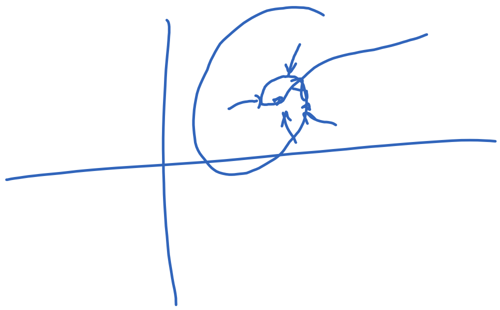
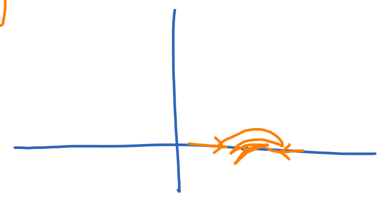
# Lecture 13

Monday, April 27, 2020 8:56 AM

$f$ is cont. at $z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$





$$f(z) = \underbrace{u(z)}_{\mathrm{Re} f(z)} + i \underbrace{v(z)}_{\mathrm{Im}\, f(z)}$$

$$f(z_0) = u(z_0) + i v(z_0)$$

$$\underbrace{\lim_{z \to z_0} f(z) = \underbrace{f(z_0)}_{L}}_{f \text{ is cont at } z_0} \quad \overset{\text{equiv.}}{\Longleftrightarrow} \quad \underbrace{\begin{cases} \lim_{z \to z_0} u(z) = u(z_0) \\ \lim_{z \to z_0} v(z) = v(z_0) \end{cases}}_{u, v \text{ are cont at } z_0.}$$

Ex: $f(z) = z^2$

$z = x + iy$

$$f(z) = (x+iy)^2 = \underbrace{x^2 - y^2}_{u(z)} + i \underbrace{2xy}_{v(z)}$$

$u$ can be considered as $u(x,y) = x^2 - y^2$
$v$ " " $v(x,y) = 2xy.$ } cont. functions

Conclude: $f(z) = z^2$ is cont. at every $z$.

Ex: $f(z) = \underbrace{|z|}_{\text{real}} = u(z) + i\,v(z)$

$$\begin{cases} u(z) = |z| \rightsquigarrow u(x,y) = |x+iy| = \sqrt{x^2+y^2} \\ v(z) = 0 \end{cases}$$

$u(x,y) = \sqrt{x^2+y^2}$ continuous. (Math 254)

Ex: $f(z) = \dfrac{\bar z}{|z|} = u(z) + i\,v(z)$.

$z = x+iy$

$\bar z = x - iy$

$|z| = \sqrt{x^2+y^2}$

$$f(z) = \frac{x-iy}{\sqrt{x^2+y^2}} = \underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{u(x,y)} + i\underbrace{\left(\frac{-y}{\sqrt{x^2+y^2}}\right)}_{v(x,y)}$$

$u$ and $v$ are cont at any point $(x,y) \neq (0,0)$

$$u(x,y) = \frac{x}{\sqrt{x^2+y^2}}$$

$(x_n, y_n) = \left(\frac{1}{n}, 0\right)$ --- $u(x_n,y_n) = \dfrac{x_n}{\sqrt{x_n^2+y_n^2}} = \dfrac{1/n}{1/n} = 1.$

$(x_n, y_n) = \left(-\frac{1}{n}, 0\right)$ ---- $u(x_n,y_n) = \dfrac{-1/n}{1/n} = -1$

$u$ is dis. at $(0,0)$.

$\to$ $f$ is dis. at $z=0$.

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Ex: $f(z) = z$

$z = x+iy$

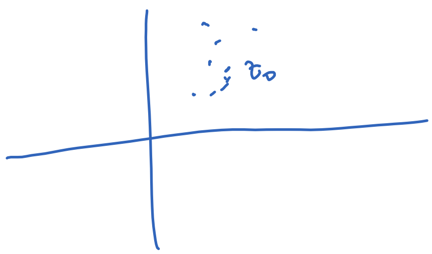
$f(z) = x+iy$

$$\begin{cases} u(x,y) = x \\ v(x,y) = y \end{cases} \to \text{cont.}$$

Math 254

Fix $z_0 \in \mathbb{C}$.

Let $(z_n)$ be a sequence that converges to $z_0$.




$f(z_n) = z_n \to z_0 = f(z_0)$

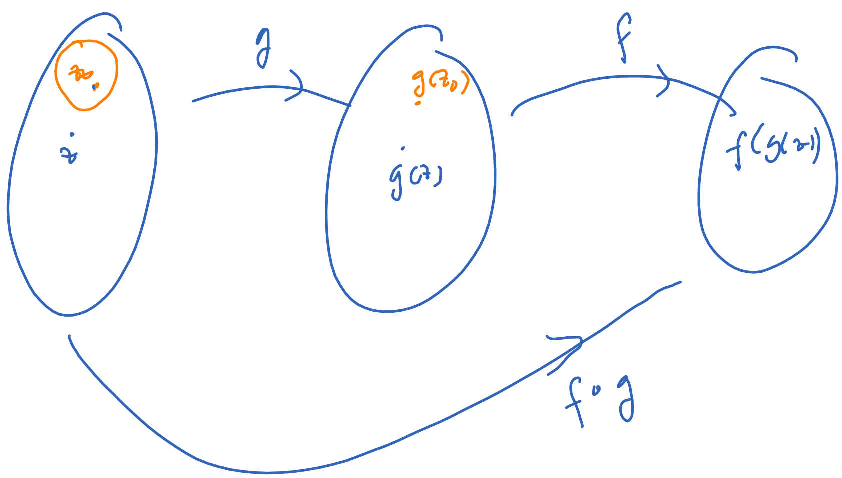
$f$ is cont. at $z_0$.

**Rule:** If $f$ & $g$ are cont. at $z_0$ then $f+g$ is cont. at $z_0$

" " " " $fg$ " "

" " " and $g(z_0) \neq 0$ then $\frac{f}{g}$ is cont at $z_0$.

If $\begin{cases} g \text{ is cont at } z_0 \\ f \text{ is cont at } g(z_0) \end{cases}$ then $f \circ g$ is cont at $z_0$.




$f(z) = z$ ... continuous

$f(z) = c$ (constant) ... continuous

$\underline{z^3} + z + 1$ ... continuous (at any point)

$z \cdot z \cdot z$

Any poly. is a cont. function

$\dfrac{P(z)}{Q(z)}$ ... rational function is cont. at points where $Q(z) \neq 0$.

Ex: $\dfrac{z}{z-1}$ is cont. on $\mathbb{C} \setminus \{1\}$.

$\dfrac{z^2+1}{z-i}$ cont. on $\mathbb{C}$.

$$\parallel$$

$$\frac{z^2 - i^2}{z-i} = \frac{(z-i)(z+i)}{z-i} = \underline{z+i} \quad \text{cont.}$$

Ex: $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$

$z = x+iy$ $\quad = \underbrace{e^x \cos y}_{u(x,y)} + i \underbrace{e^x \sin y}_{v(x,y)}$

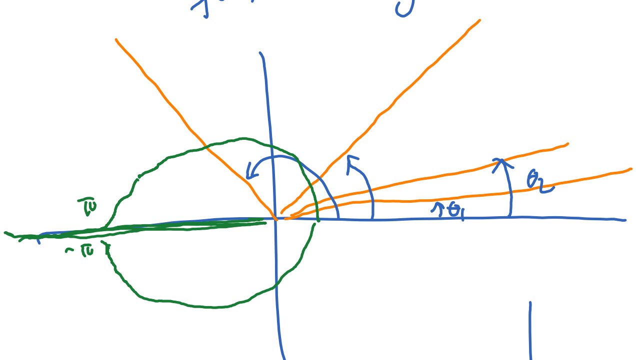
$e^z$ is cont.

$\sin z$ ...

$\cos z$ ...

Ex:

$f(z) = \operatorname{Arg} z \in (-\pi, \pi]$

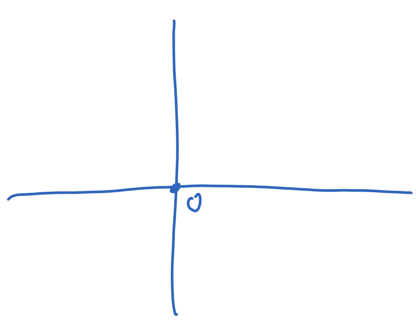


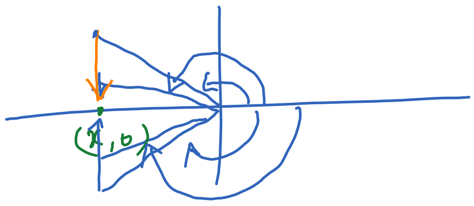

$f$ is discont. on $\mathbb{R}_{<0}$.

$f$ is not def. at $z=0$.

Conclude:

$f$ is discont on $\mathbb{R}_{\leq 0}$.






$x < 0.$

We want to show that $f(z) = \mathrm{Arg}(z)$ is dis. at $(x,0)$.

Consider $z_n = x + i\frac{1}{n}$.

$f(z_n) = \mathrm{Arg}\left(x + i\frac{1}{n}\right) \longrightarrow \pi.$

$z_n = x - i\frac{1}{n}$

$f(z_n) = \mathrm{Arg}\left(x - i\frac{1}{n}\right) \longrightarrow -\pi$

$\neq$

Food for thought:

$f(z) = \mathrm{Log}\, z$ ... where is $f$ cont?

$f(z) = \sqrt{iz - 1}$ ... " " "

principal logarithm.

$$\sqrt{z} = e^{\frac{1}{2} \mathrm{Log}\, z}$$