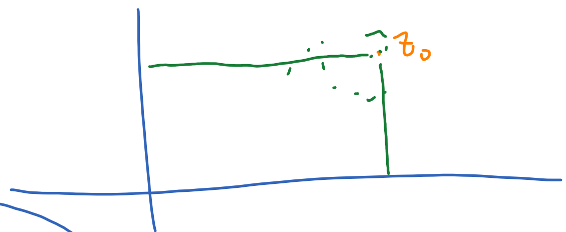
Lecture 12

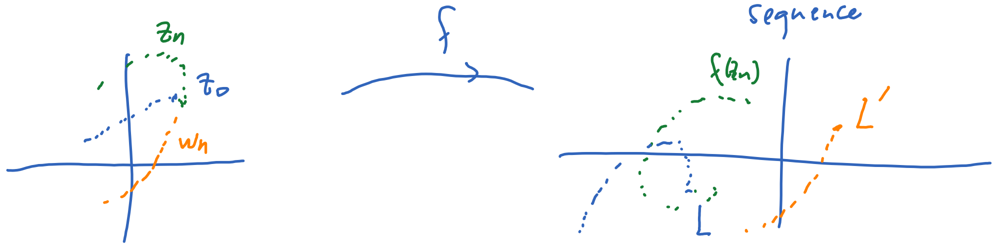
Friday, April 24, 2020 8:35 AM

$\lim\limits_{n\to\infty} z_n = z_0 \quad \overset{def}{\rightsquigarrow} \quad |z_n - z_0| \to 0$ as $n \to \infty$

Equiv. $\begin{cases} \operatorname{Re} z_n \to \operatorname{Re} z_0 \\ \operatorname{Im} z_n \to \operatorname{Im} z_0 \end{cases}$

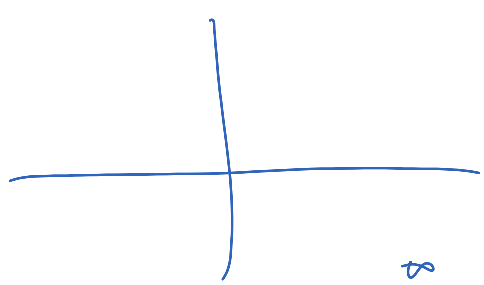


$\lim\limits_{z\to z_0} f(z) = L \quad \longrightarrow \quad$ For any seq $z_n \to z_0$, we have $\underbrace{f(z_n)}_{\text{sequence}} \to L$



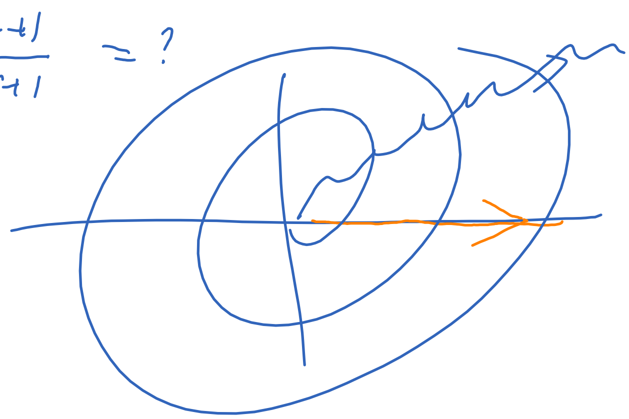
z_n

$\lim z_n = \infty \quad \overset{def}{\rightsquigarrow} \quad |z_n| \to \infty$



$\lim\limits_{z\to z_0} f(z) = \infty \quad \overset{def}{\rightsquigarrow} \quad$ For any seq $z_n \to z_0$, we have $\underbrace{|f(z_n)| \to \infty}_{f(z_n) \to \infty}$.

$$\lim_{z\to\infty} \frac{z+1}{z^2+1} = ?$$



$$\frac{x+1}{x^2+1} \to 0$$

$$\frac{z+1}{z^2+1} = \frac{\frac{1}{z}+\frac{1}{z^2}}{1+\frac{1}{z^2}} \longrightarrow \frac{0+0}{1+0} = 0.$$

$\frac{1}{z} \to 0$ as $z \to \infty$?

Take a seq. $z_n \to \infty$ we see that

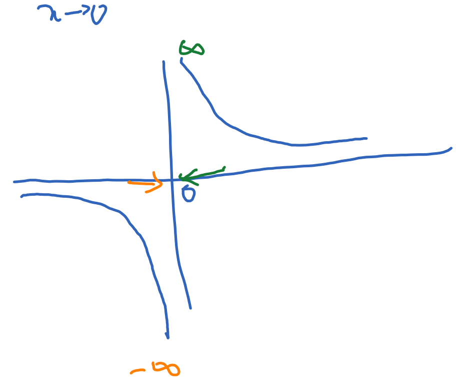
$$\left| \frac{1}{z_n} - 0 \right| = \frac{1}{|z_n|} \to 0$$

we have $\frac{1}{z_n} \to 0$.

$$\frac{1}{z} \to 0 \qquad \frac{1}{z^2} = \frac{1}{z}\cdot\frac{1}{z} \to 0\cdot 0 = 0$$

$$\lim_{z\to i} \frac{z+1}{z^2+1} = ?$$

$\lim_{x\to 0} \frac{1}{x}$ DNE



$$\frac{z+1}{z^2+1} = \frac{z+1}{(z-i)(z+i)}$$

$z^2+1 = z^2 - i^2$

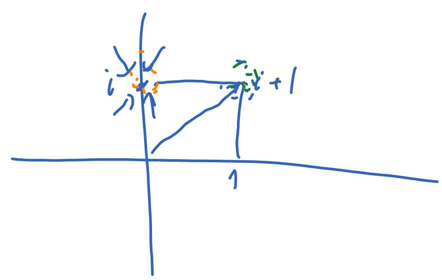
$$= \frac{\frac{z+1}{z+i}}{z-i}$$

$$\frac{z+1}{z^2+1} = \frac{\overbrace{\dfrac{z+1}{z+i}}^{\to \frac{i+1}{i+i}}}{\underbrace{z-i}_{\to 0}} \xrightarrow{\text{guess}} \infty$$

Take $z_n \to i$. want to see if $\dfrac{z_n+1}{z_n^2+1} \to \infty$

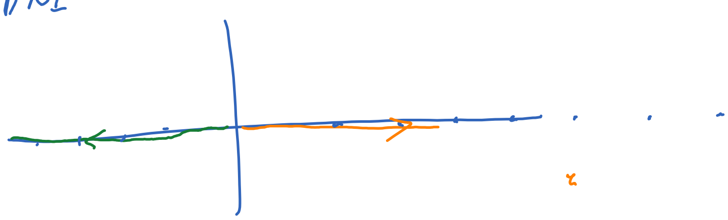
$$\underbrace{\frac{z_n+1}{z_n^2+1}}_{\displaystyle \frac{\frac{z_n+1}{z_n+i}}{z_n-i}}$$

$$\left|\frac{z_n+1}{z_n^2+1}\right| = \frac{\left|\frac{z_n+1}{z_n+i}\right|}{|z_n-i|} = \frac{\overbrace{\dfrac{|z_n+1|}{|z_n+i|}}^{\to \frac{|i+1|}{|i+i|} = \frac{\sqrt{2}}{2}}}{\underbrace{|z_n-i|}_{\to 0^+}} \longrightarrow \infty$$



Ex:

$$\lim_{z\to\infty} e^z = \ ? \leftarrow \text{DNE}$$

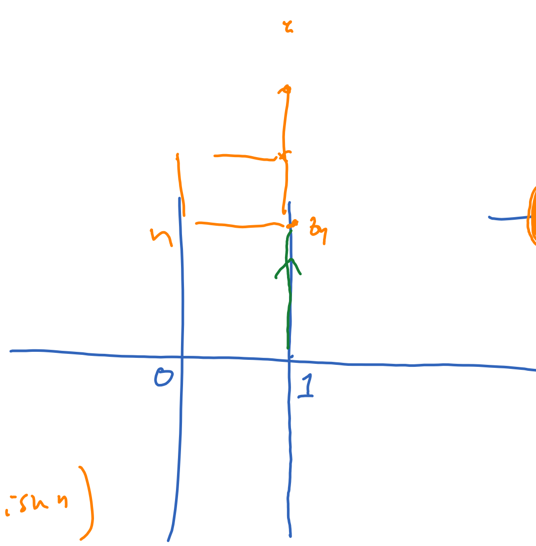


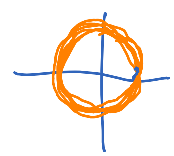
$z_n = n$

$$e^{z_n} = e^n \longrightarrow \infty$$

$z_n = -n$

$$e^{z_n} = e^{-n} \longrightarrow 0$$





$z_n = 1 + in$

$$e^{z_n} = e^{1+in} = e(\cos n + i \sin n)$$

$$\lim_{z\to z_0} f(z) = L \quad \xleftrightarrow{\text{equiv.}} \quad \begin{cases} \lim \operatorname{Re} f(z) = \operatorname{Re} L \\ \lim \operatorname{Im} f(z) = \operatorname{Im} L \end{cases}$$

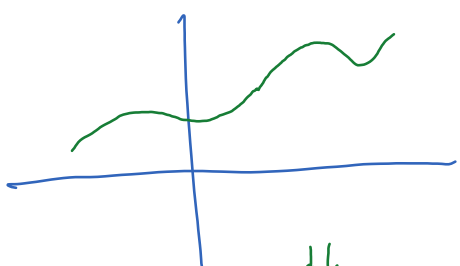
$$\underbrace{f(z) = u(z) + i\,v(z)}\ \longrightarrow \text{standard form}$$

$$L = l_1 + i\,l_2$$

$$\lim f(z) = L \quad \text{iff} \quad \begin{cases} u(z) \to l_1 \\ v(z) \to l_2 \end{cases} \quad \text{as } z \to z_0.$$

Def: A function f is said to be continuous at z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0)$$



real variable

Equiv.

$$\begin{cases} u(z) \text{ is cont at } z_0 \\ v(z) \text{ is cont at } z_0 \end{cases}$$

$\operatorname{Arg} z$

$\operatorname{Log} z$

...

$e^{z} = -1 \rightsquigarrow z = \log(-1) = \ln 1 + i(\pi + k2\pi)$

what is log?

$e^{z} = \underbrace{e^{x}}\underbrace{e^{iy}}$

If $e^{z} = w$ then $\log w = z$. def

$e^{z_0} = w$

$e^{z + i2\pi} = w$

$\underbrace{2}^{z} = 4 \leftarrow z \ln 2 = \ln 2$

$e^{z\log 2} = 4$

$z \log 2 = \log 4$

$z = \dfrac{\log 4}{\log 2}$ $\leftarrow k$, $\nwarrow \ell$

$z = \arcsin(1-2i) = \dfrac{1}{i}\log\left(\underbrace{i(1-2i)}_{i+2} + \underbrace{\sqrt{1-(1-2i)^2}}_{\sqrt{4+4i}}\right)$

$= \dfrac{1}{i}\log\left(i+2+\sqrt{4+4i}\right)$

$\sqrt{4+4i} = \sqrt{\underbrace{\sqrt{32}\, e^{i\frac{\pi}{4}}}_{w}} = \underbrace{\pm\sqrt[4]{32}}_{\text{red}}\, e^{i\frac{\pi}{8}}$ ①

②

$\sqrt{w} = e^{\frac{1}{2}\log w} = e^{\frac{1}{2}\left(\ln\sqrt{32} + i\left(\frac{\pi}{4} + k2\pi\right)\right)} = e^{\frac{1}{2}\ln\sqrt{32}}\, e^{i\left(\frac{\pi}{8} + k\pi\right)}$

$$\sqrt{w} = e^{\cdots}$$

$$= e^{\frac{1}{2}\left(\ln\sqrt{32} + i\left(\frac{\pi}{4} + k2\pi\right)\right)} = \underbrace{e^{\frac{1}{2}\ln\sqrt{32}}}_{\sqrt[4]{32}} \; \underbrace{e^{i\left(\frac{\pi}{8} + k\pi\right)}}_{e^{i\frac{\pi}{8}}\, e^{ik\pi}} \quad \pm 1$$

$$\frac{1}{i}\log\left(i+2+\sqrt{4+4i}\right)$$

Plus sign. $\frac{1}{i}\log\left(\underbrace{i+2+\sqrt[4]{32}\,e^{i\frac{\pi}{8}}}\right)$

$$i+2+\sqrt[4]{32}\cos\frac{\pi}{8}+i\sqrt[4]{32}\sin\frac{\pi}{8}$$

$$=\left(\underbrace{2+\sqrt[4]{32}\cos\frac{\pi}{8}}\right)+i\left(\underbrace{1+\sqrt[4]{32}\sin\frac{\pi}{8}}\right)$$

$$= r e^{i\theta}$$