Lecture 10

Monday, April 20, 2020 8:42 AM

$$\underbrace{\ln,\ \log,\ \text{Log}}_{\text{base } e}$$

e: 2.71...

$$\underbrace{\log_2}\, 3 = \frac{\ln 3}{\ln 2}$$

$a : \quad 2^a = 3$

Solve for z such that $\boxed{2^z = 3}$

$$2^z = e^{z\log 2} \qquad \text{(by def)} \qquad z^a = e^{a\log z}.$$

$$\log z = \ln|z| + i\arg(z) \longrightarrow \text{logarithm}$$

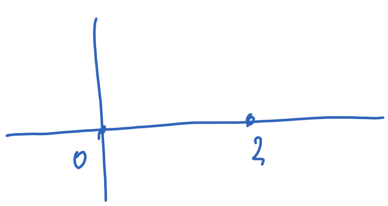
multi-valued

$$\text{Log}\, z = \ln|z| + i\,\text{Arg}(z) \longrightarrow \begin{cases}\text{principal logarithm} \\ \text{principal branch of the logarithm.}\end{cases}$$

principal argument

$(-\pi, \pi]$

$$\log 2 = \ln 2 + i(0 + k2\pi) = \ln 2 + ik2\pi \qquad (k \in \mathbb{Z})$$



$$2^z = e^{z(\ln 2 + ik2\pi)}$$

Solve for z:

$$e^{z(\ln 2 + ik2\pi)} = 3$$

$$\rightsquigarrow z(\ln 2 + ik2\pi) = \log 3$$

$$z(\ln 2 + ik2\pi) = \underbrace{\log 3}_{\ln 3 + il2\pi} \qquad k, l \in \mathbb{Z}$$

$$\boxed{z = \frac{\ln 3 + il2\pi}{\ln 2 + ik2\pi}}$$

There's no such thing as $\underbrace{\log_z 3}$ for complex variable.

$\underbrace{\log, \text{Log}, \ln}_{\text{base } e}$

$$z^a \underset{=}{\text{def}} e^{a\log z}$$

$$z \xrightarrow{\log} \log z \xrightarrow{\times a} a\log z \xrightarrow{\exp} e^{a\log z} = z^a$$

multivalue — single — single

$$e^i = \exp(i) = \cos 1 + i\sin 1$$

when $z = e$, we know what e^a is, without going into the chain

$$e^i \qquad e^{i\log e} = e^{i(\ln e + ik2\pi)} = e^{i(1 + ik2\pi)} = e^{i - k2\pi}$$

$$= e^{-k2\pi}(\cos 1 + i\sin 1)$$

$$\underline{e^z} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

$$z^2 = e^{2\log z} = e^{2(\ln r + i(\theta + k2\pi))} = e^{2\ln r + i(2\theta + k2\pi)}$$

$$= \underbrace{e^{2\ln r}}\, e^{i(2\theta + k2\pi)}$$

$$z = re^{i\theta}$$

$$= r^2 e^{i2\theta} \underbrace{e^{ik2\pi}}_{e^0 = 1}$$

$$= r^2 e^{i2\theta}$$

$z^a = e^{a\log z}$ .... used when $\begin{cases} z \neq e \\ a \text{ is not integer.} \end{cases}$

$e^{x+y} = e^x e^y$ ... knows

$e^{u+v} = e^u e^v$ .... true for $u, v \in \mathbb{C}$

$z^{u+v} \neq z^u z^v$ (in general)

$\underbrace{2^{1+i}} \neq 2^1 2^i$

$e^{(1+i)\log 2} \neq e^{\ln 2} e^{i\log 2}$

$(1+i)(\ln 2 + ik2\pi) \overset{?}{=} \underbrace{\ln 2} + i\log 2$

$(zw)^a \neq z^a w^a$ (in general)

$e^{a\log(zw)}$ $\quad\quad \underbrace{e^{a\log z}\, e^{a\log w}}$

$e^{a(\log z + \log w)}$

$\underbrace{\log(zw)} \neq \underbrace{\log z} + \underbrace{\log w}$

$z = 1$ $\quad\quad$ Set $\quad\quad$ Set $= \ln 1 + i\left(\frac{\pi}{2} + l2\pi\right)$

$w = i$ $\quad\quad$ $\| $ $\quad\quad 0 + 1 + ik2\pi$

$z = 1$

$w = i$

$\underbrace{}_{i\left(\frac{\pi}{2} + m2\pi\right)}$

$\ln 1 + ik2\pi$

$\underbrace{\ln 1 + i\left(\frac{\pi}{2} + l2\pi\right)}_{i\left(\frac{\pi}{2} + (k+l)2\pi\right)}$

$$\underbrace{\mathrm{Log}(zw)}_{(-\pi,\pi]} \neq \underbrace{\mathrm{Log}\, z}_{} + \underbrace{\mathrm{Log}\, w}_{}$$

$\mathrm{Arg}\, z \in (-\pi, \pi]$ $\quad$ $\mathrm{Arg}\, w \in (-\pi, \pi]$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$e^z \; \cdots\cdots \; \log z$

$\arcsin z = ?$

$\underline{\arcsin z} = w$ ----- means $\sin w = z$

How to find w given z?

$$\sin w = z$$

$$\frac{e^{iw} - e^{-iw}}{2i} = z$$

$\underline{u = e^{iw}}$

$$\frac{u - \frac{1}{u}}{2i} = z$$

$$\frac{u^2 - 1}{u} = 2iz$$

$$u^2 - 2iz\,u - 1 = 0$$

$$u = iz + \sqrt{(iz)^2 + 1}$$

$$e^{iw} = iz + \sqrt{1 - z^2}$$

$$iw = \log\left(iz + \sqrt{1 - z^2}\right)$$

$$w = \frac{1}{i} \log\left(iz + \sqrt{1 - z^2}\right)$$

$$\arcsin z = \frac{1}{i} \log\left(iz + \sqrt{1 - z^2}\right)$$

arcsin

log

Log single value

$$\operatorname{Arcsin} z = \frac{1}{i}\operatorname{Log}\left(iz + \sqrt{1-z^2}\right)$$

principal

$$\sqrt{1-z^2} = (1-z^2)^{1/2} = e^{\frac{1}{2}\operatorname{Log}(1-z^2)}$$

$\sqrt{-3}$: $i\sqrt{3}$, $-i\sqrt{3}$

$$\frac{1}{i}\log\left(\frac{i}{2} \pm \sqrt{\frac{3}{4}}\right)$$

- plus sign:

$$\frac{1}{i}\log\left(\frac{i}{2} + \frac{\sqrt{3}}{2}\right) = \frac{1}{i}\left(\underbrace{\ln 1}_{0} + i\left(2k\pi + \frac{\pi}{6}\right)\right)$$

$$= 2k\pi + \frac{\pi}{6}.$$

- minus:

$$\frac{1}{i}\log\left(\frac{i}{2} - \frac{\sqrt{3}}{2}\right) = \frac{1}{i}\left(\ln 1 + i\left(\frac{5\pi}{6} + 2j\pi\right)\right)$$

$$= \frac{5\pi}{6} + 2j\pi$$

$$\arcsin\left(\frac{1}{i}\right) = \underbrace{\left\{2k\pi + \frac{\pi}{6},\ k \in \mathbb{Z}\right\}} \cup \underbrace{\left\{\frac{5\pi}{6} + 2j\pi,\ j \in \mathbb{Z}\right\}}$$

$$\sqrt{2} = \{\sqrt{2}, -\sqrt{2}\}$$

complex

$$\pm\sqrt{\frac{3}{4}}$$