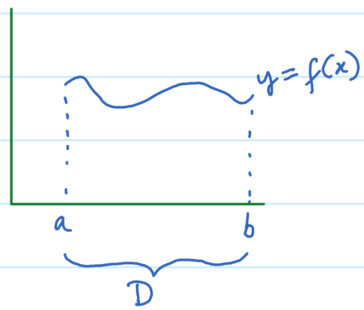
Lecture 1

Monday, March 30, 2020

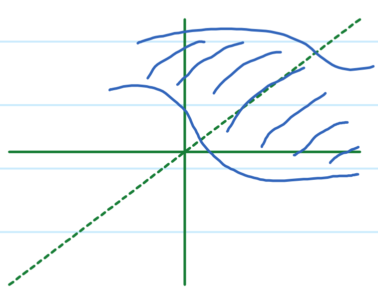
In Math 251 and 252, we learn Calculus (i.e. how to calculate) functions of one variable. More specifically, we learned how to compute limits, derivatives and integrals. Geometrically, a function $f: D \subset \mathbb{R} \to \mathbb{R}$ can be understood as a curve (the graph of f).



The derivative of f reflects the slope of the tangent line along the curve. The integral of f represents the area under the curve.

Therefore, although the Calculus of one real-variabled functions can be done purely analytically, the results are meaningful geometrically.

In Math 254, we learned how to do Calculus on multivariable functions $f: D \subset \mathbb{R}^n \to \mathbb{R}^m$. We defined derivatives along each direction (i.e. directional derivative). We defined double integral, triple integral, multiple integral, iterated integral, .. A function from $D \subset \mathbb{R}^2$ to $\mathbb{R}$ can be viewed as a surface in 3D-space (its graph).



Directional derivative is the slope of a cross section of a surface. Integral is the volume under the surface.

In this course, we will learn how to do Calculus on functions of complex variables, i.e. functions $f: D \subset \mathbb{C} \to \mathbb{C}$. One can use many techniques from Math 251, 252, 254 to study such functions. The new elements do not lie in the fact that the function has complex values, but rather the fact that the function has complex variables. In fact, one doesn't need to learn any new techniques other than that of Math 251, 252, 254 to do calculus on functions from $\mathbb{R}$ to $\mathbb{C}$.

The fact that the variable belongs to $\mathbb{C}$ causes the main difference. For example, to concept of limit has to be considered more carefully.

If the Calculus of one real-variable and multiple real-variables helps us calculate physical quantities such as rate of change, length of curve, area, volume of shapes, ... then what are the roles of the Calculus of complex variables?

How to do Calculus on functions of complex variables is, on one hand, a natural question coming out of curiosity. On the other hand, it helps solve challenging problems that only involve real variables. For example, by the Residue theorem which we will learn, we will be able to compute:

$$\int_0^\infty \frac{1}{x^3+1}\,dx = \frac{2\pi}{3\sqrt{3}},$$

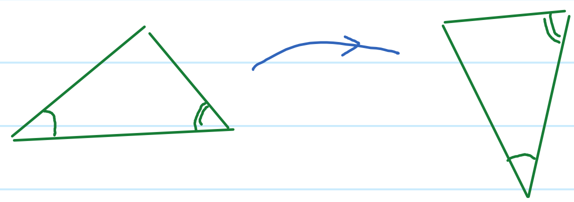
$$\int_0^\infty \frac{1}{x^4+1}\,dx = \frac{\sqrt{2}}{4}\pi,$$

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

It is surprising how number π could arise in these integrals. Functions with complex variables are introduced to turn those integrals into line integrals on the complex plane. In this way, complex variables play an intermediate role, helping us solve the problem. Looking at the identities above, one sees no trace of complex variables.

This phenomenon is analogous to the story of a dying father who wants to distribute his property to his three sons. He has 17 cows. He wants to give the first son a half of the cows he has, the second son a third of the cows, and his third son a nineth of the cows. Seeing that it is impossible to do so with 17 cows, he borrows a cow from his neighbor, making his total number of cows 18. Then he gives 9 cows to the first son, 6 cows to the second son, and 2 cows to the third son. There is still one cow left. He returns it to the neighbor.

Another application of Calculus of complex variables is that differentiable functions (which we will later call holomorphic functions) are angle-preserving transformations on the plane. This is a useful class of transformations, called conformal mappings. The rotation and scaling are examples of conformal mappings because they preserve the angles.



We will visualize geometric properties of complex variabled functions by a software called Mathematica. It is available under OSU license.