MTH 483/583 Complex Variables – Homework 4

Solution Key

Spring 2020

Exercise 1. Find the following limits. Distinguish between the limit that is equal to ∞ and limit that does not exist. If the limit is a complex number, write your answers in either standard or polar form.

(a) $\lim_{z \to i+1} \frac{(z-i)(2z+1)}{z+1}$

Answer: The function

$$f(z) = \frac{(z-i)(2z+1)}{z+1}$$

is a rational function. It is continuous everywhere except at $i+1$. Therefore,

$$\lim_{z \to i+1} f(z) = f(i+1) = \frac{(i+1-i)(2i+3)}{i+2}$$

$$= \boxed{\frac{8}{5} + i\frac{1}{5}}$$

□

(b) $\lim_{z \to i} \frac{z^2+1}{(z-i)(z+1)}$

Answer:

$$\lim_{z \to i} \frac{z^2+1}{(z-i)(z+1)} = \lim_{z \to i} \frac{(z-i)(z+i)}{(z-i)(z+1)}$$

$$= \lim_{z \to i} \frac{z+i}{z+1} = \frac{2i}{i+1}$$

$$= \boxed{1+i}$$

□

(c) $\lim_{z \to -i} \frac{z^3-i}{(z+i)(z-1)}$

Answer: We see that the limit is currently in the indefinite form $\frac{0}{0}$. We can factor $(z+i)$ from $z^3 - i$ by using long division: $z^3 - i = (z+i)(z^2 - iz - 1)$. Thus,

$$\lim_{z\to -i} \frac{z^3 - i}{(z+i)(z-1)} = \lim_{z\to -i} \frac{(z+i)(z^2 - iz - 1)}{(z+i)(z-1)}$$

$$= \lim_{z\to -i} \frac{z^2 - iz - 1}{z - 1} = \frac{(-i)^2 - i(-i) - 1}{-1 - i}$$

$$= \boxed{\frac{3}{2} - i\frac{3}{2}}$$

□

(d) $\lim_{z\to e^{i\frac{\pi}{3}}} \frac{z}{z^3+1}$

Answer: Because z approaches a nonzero number, namely, $e^{i\frac{\pi}{3}}$, and the denominator approaches

$$(e^{i\frac{\pi}{3}})^3 + 1 = e^{i\pi} + 1 = -1 + 1 = 0$$

the limit of the fraction is equal to $\boxed{\infty}$.

□

(e) $\lim_{z\to\infty} \frac{z^2+1}{(z-2i)(z+1)}$

Answer:

$$\lim_{z\to\infty} \frac{z^2+1}{(z-2i)(z+1)} = \lim_{z\to\infty} \frac{1+\frac{1}{z^2}}{(1-\frac{2i}{z})(1+\frac{1}{z})} = \frac{1+0}{(1-0)(1+0)} = \boxed{1}$$

□

(f) $\lim_{z\to\infty} \frac{z^2+1}{z+1}$

Answer:

$$\lim_{z\to\infty} \frac{z^2+1}{z+1} = \lim_{z\to\infty} \frac{1+\frac{1}{z^2}}{\frac{1}{z}+\frac{1}{z^2}}$$

The numerator goes to 1 (a nonzero number) and the denominator goes to 0. Hence, the limit is equal to ∞. □

(g) $\lim_{z\to\infty} \frac{\bar{z}}{z}$

Answer: Let us analyse the problem as follows. Write $z = re^{i\theta}$. Taking z to ∞ is equivalent to taking r to ∞. There is no constraint on the argument θ as $z \to \infty$. We have

$$\frac{\bar{z}}{z} = \frac{re^{-i\theta}}{re^{i\theta}} = e^{-2i\theta}$$

This number is independent of r. If z goes to infinity along the half-line where the argument is θ then the limit is equal to $e^{-2i\theta}$. This limit is different when θ varies. Thus, the limit does not exist. To make our reasoning more rigorous, let us consider two sequences: $z_n = n$ (tending to ∞ along the positive real axis) and $w_n = in$ (tending to ∞ along the positive imaginary axis). We have

$$f(z_n) = f(n) = \frac{n}{n} = 1$$

and

$$f(w_n) = f(in) = \frac{-in}{in} = -1$$

We see that $\lim f(z_n) = 1 \neq \lim f(w_n) = -1$.

□

(h) $\lim_{z\to\infty} e^{-1/z^2}$

Answer: When $z \to \infty$, we have $\frac{1}{z^2} \to 0$. This is because

$$\left|\frac{1}{z^2} - 0\right| = \left|\frac{1}{z^2}\right| = \frac{1}{|z|^2} \to 0.$$

Because the exponential is a continuous function, we have

$$\lim_{z\to\infty} e^{-1/z^2} = e^0 = \boxed{1}$$

□

(i) $\lim_{z\to\infty} \sin z$

Answer: Let $z_n = 2n\pi + \frac{\pi}{2}$ and $w_n = 2n\pi + \frac{3\pi}{2}$. Both sequences goes to infinity. However,

$$\sin(z_n) = 1$$

$$\sin(w_n) = -1$$

Because the limits of $\sin z_n$ and of $\sin w_n$ are different, the limit $\lim_{z\to\infty} \sin z$ $\boxed{\text{does not exist}}$.

□

(j) $\lim_{z\to\infty} \frac{\sin z}{z}$

Answer: Let $z_n = (n + 1/2)\pi$. Then

$$\frac{\sin z_n}{z_n} = \frac{1}{z_n} \to 0$$

because $z_n \to \infty$.

On the other hand, let $w_n = in$. Then

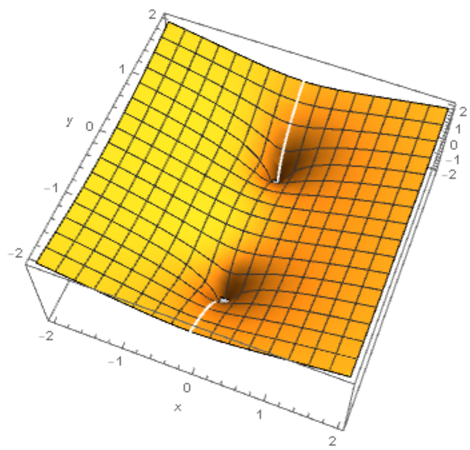
$$\frac{\sin w_n}{w_n} = \frac{e^{-n} - e^n}{-2n} \to \frac{e^n - e^{-n}}{2n} = \infty$$

Therefore, the limit $\boxed{\textbf{does not exist}}$.

□

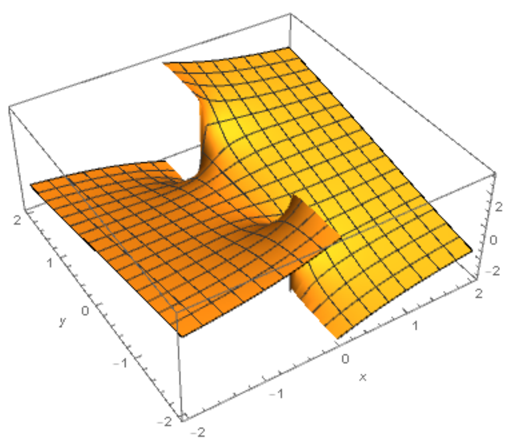
Exercise 2. (a) To get some visual insights about function f, let us plot the real part and imaginary part of f.

```
Plot3D[Re[Log[(x + I*y)^2 + 1]], {x, -2, 2}, {y, -2, 2},
 AxesLabel -> Automatic]
```



Graph of the real part of $\text{Log}(z^2 + 1)$

```
Plot3D[Im[Log[(x + I*y)^2 + 1]], {x, -2, 2}, {y, -2, 2},
 AxesLabel -> Automatic]
```



Graph of the imaginary part of $\text{Log}(z^2 + 1)$

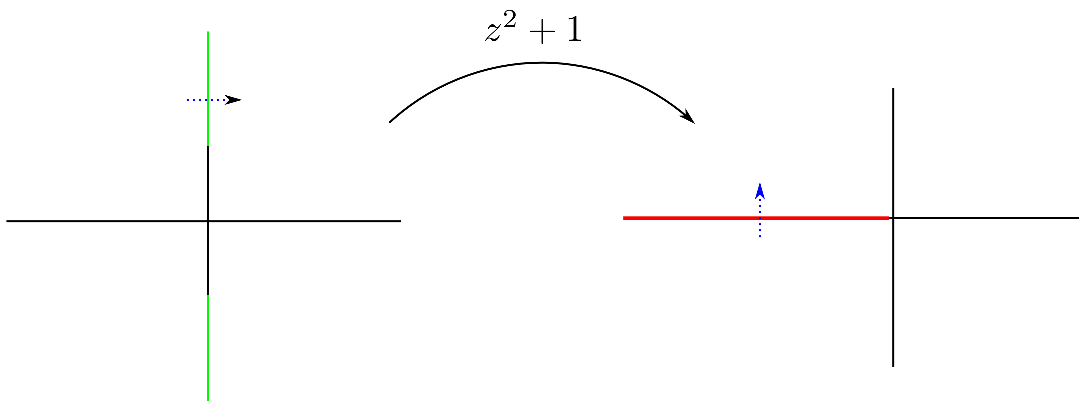
From the graphs, we observe that f is discontinuous on the lines $z = ti$ with $t \leq -1$ or $t \geq 1$. Now let us make verify this guess. The principal logarithm is discontinuous on the negative real line $\mathbb{R}_{\leq 0}$. Thus, f is continuous at any point z where $z^2 + 1 \notin \mathbb{R}_{\leq 0}$. To find all z such that $z^2 + 1 \in \mathbb{R}_{\leq 0}$, we write $z = x + iy$. Then

$$z^2 + 1 = (x^2 - y^2 + 1) + i2xy.$$

For this number to be on $\mathbb{R}_{\leq 0}$, we need

$$\begin{cases} x^2 - y^2 + 1 \leq 0, \\ 2xy = 0. \end{cases}$$

We get $x = 0$ and $y^2 \geq 1$. Hence, the set of all z such that $z^2 + 1 \in \mathbb{R}_{\leq 0}$ is the lines $z = iy$ with $y \leq -1$ or $y \geq 1$. We will show that f is indeed discontinuous at every point on these lines.

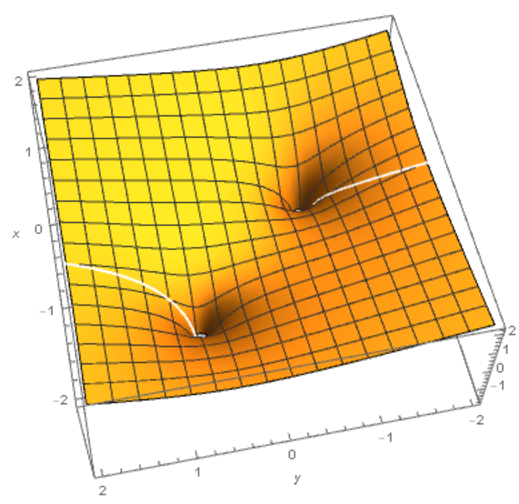


Discontinuity of $\text{Log}(z^2 + 1)$

When z crosses the green line, z^2+1 crosses the red line, causing $\text{Log}(z^2+1)$ to jump by $2\pi i$.

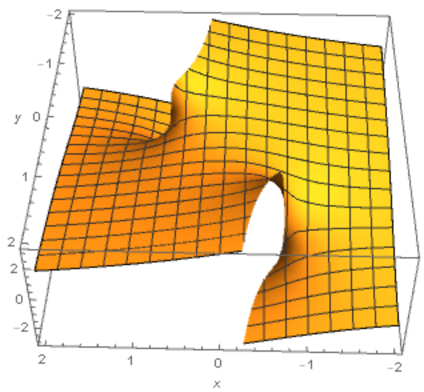
(b) To get some visual insights about function f, let us plot the real part and imaginary part of f.

```
Plot3D[Re[Log[(x + I*y)^2 + I]], {x, -2, 2}, {y, -2, 2},
 AxesLabel -> Automatic]
```



Graph of the real part of $\text{Log}(z^2 + i)$

```
Plot3D[Im[Log[(x + I*y)^2 + I]], {x, -2, 2}, {y, -2, 2},
 AxesLabel -> Automatic]
```



Graph of the imaginary part of $\mathrm{Log}(z^2+i)$

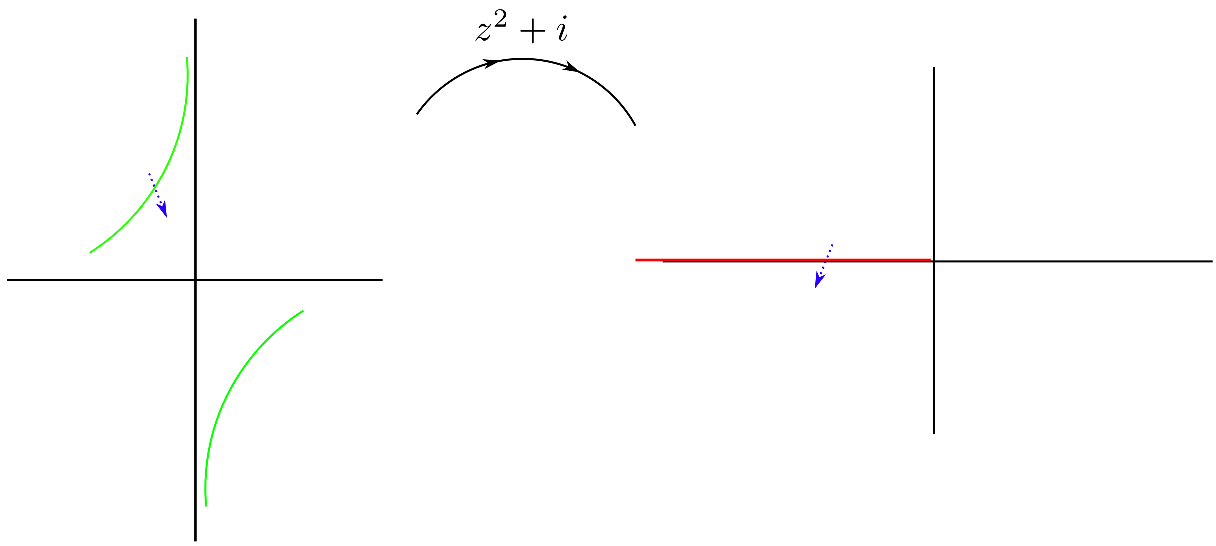
From the graphs, we observe that f is discontinuous on two curves. It is not clear yet what these curves are. Let us do further analysis. The principal logarithm is discontinuous on the negative real line $\mathbb{R}_{\le 0}$. Thus, f is continuous at any point z where $z^2+i \notin \mathbb{R}_{\le 0}$. To find all z such that $z^2+i \in \mathbb{R}_{\le 0}$, we write $z = x+iy$. Then

$$z^2+i = (x^2-y^2) + i(2xy+1).$$

For this number to be on $\mathbb{R}_{\le 0}$, we need

$$\begin{cases} x^2 - y^2 \le 0, \\ 2xy+1 = 0. \end{cases}$$

We get $y = -1/(2x)$ and $x^4 \le 1/4$. Hence, the set of all z such that $z^2+i \in \mathbb{R}_{\le 0}$ is the hyperbola $y = -1/(2x)$ with $-1/\sqrt{2} \le x < 0$ and $0 < x \le 1/\sqrt{2}$. We will show that f is indeed discontinuous at every point on these hyperbola.

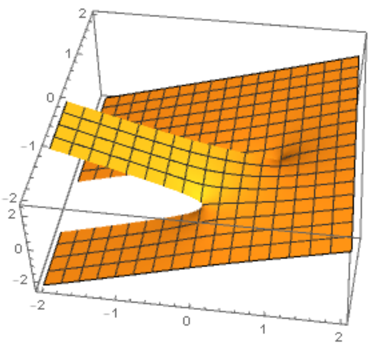


Discontinuity of $\mathrm{Log}(z^2+i)$

When z crosses the green curve, z^2+i crosses the red line, causing $\mathrm{Log}(z^2+i)$ to jump by $2\pi i$.

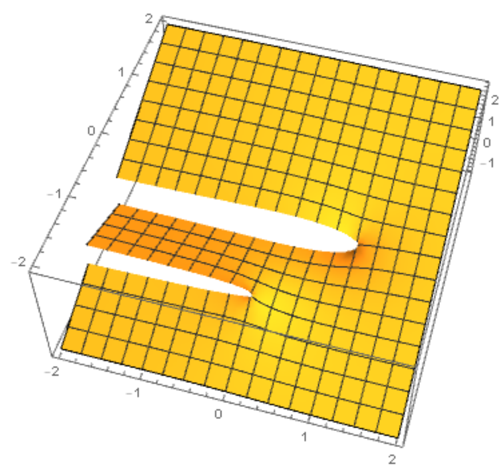
(c) To get some visual insights about function f, let us plot the real part and imaginary part of f.

```
f[z_] := Sqrt[z - 1]*Sqrt[z + I]
Plot3D[Re[f[x + I*y]], {x, -2, 2}, {y, -2, 2}, AxesLabel -> Automatic]
```



Graph of the real part of $\sqrt{z-1}\sqrt{z+i}$

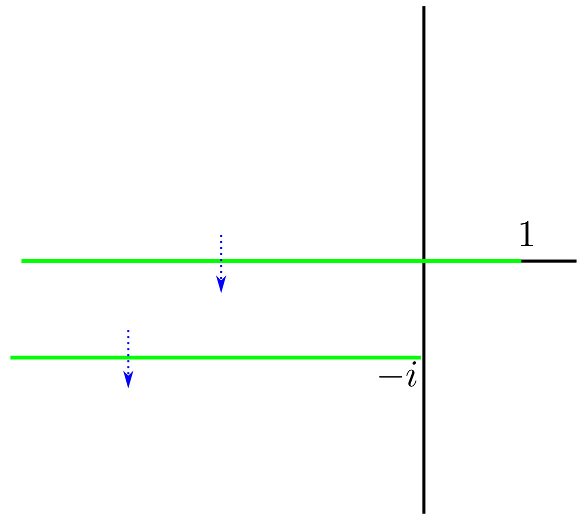
```
Plot3D[Im[f[x + I*y]], {x, -2, 2}, {y, -2, 2}, AxesLabel -> Automatic]
```



Graph of the imaginary part of $\sqrt{z-1}\sqrt{z+i}$

From the graphs, we observe that f is discontinuous on two straight lines. Let us do further analysis to determine these lines. The principal logarithm is discontinuous on the negative real line $\mathbb{R}_{\leq 0}$. Thus, f is continuous at any point z where $z-1 \notin \mathbb{R}_{\leq 0}$ and $z+i \notin \mathbb{R}_{\leq 0}$. To find all z such that $z-1 \in \mathbb{R}_{\leq 0}$, we write $z-1 = t \in \mathbb{R}_{\leq 0}$. Then $z = t+1 \in \mathbb{R}_{\leq 1}$. This is the first line. To find all z such that $z+i \in \mathbb{R}_{\leq 0}$, we write $z+i = t \in \mathbb{R}_{\leq 0}$. Then $z = t-i$. The set of all points $t-i$ where $t \in \mathbb{R}_{\leq 0}$ is the line $x \leq 0$, $y = -1$. This is the second line.

We will show that f is indeed discontinuous at every point on these lines.



Discontinuity of $\sqrt{z-1}\sqrt{z+i}$

When z crosses the curve $x \leq 1$, $y = 0$, the function $\sqrt{z-1}$ suddenly changes sign (because it is scaled by factor $e^{i\pi} = -1$) but the function $\sqrt{z+i}$ varies continuously and is nonzero. The product $\sqrt{z-1}\sqrt{z+i}$ therefore switches its sign suddenly (and is nonzero). Thus, it is discontinuous at every point on the curve $x \leq 1$, $y = 0$. One can use similar reasoning for the curve $x \leq 0$, $y = -1$.

(d)

$$f(z) = \left(z^2+1\right)^i = e^{i\mathrm{Log}(z^2+1)}$$

This is a composite function. At points z where $\mathrm{Log}(z^2+1)$ is continuous, f is also continuous. We only need to check the continuity of f at the points where $\mathrm{Log}(z^2+1)$ is discontinuous. According to Part (a), these are points lying on the lines $x = 0$, $y \geq 1$ and $x = 0$, $y \leq -1$. When z cross each line, $\mathrm{Log}(z^2+1)$ jumps by $2\pi i$. Then $\frac{1}{2}\mathrm{Log}(z^2+1)$ jumps by πi. Hence, $e^{i\mathrm{Log}(z^2+1)}$ is scaled by $e^{\pi i i} = e^{-\pi}$. That is, $f(z)$ is suddenly scaled by factor $e^{-\pi}$ when z crosses the line $x = 0$, $y \geq 1$ or the line $x = 0$, $y \leq -1$. In conclusion, the region of continuity of f is

$$\mathbb{C}\backslash\{z = x+iy : \ x = 0,\ y \leq -1 \text{ or } y \geq 1\}.$$

Exercise 3. Let $f(z) = \sqrt[3]{z}$ with the principal logarithm being used.

(a) Plot the real and imaginary part of $f(z)$.

Answer: Notice that

$$\sqrt[3]{z} = e^{\frac{1}{3}Log(z)}$$

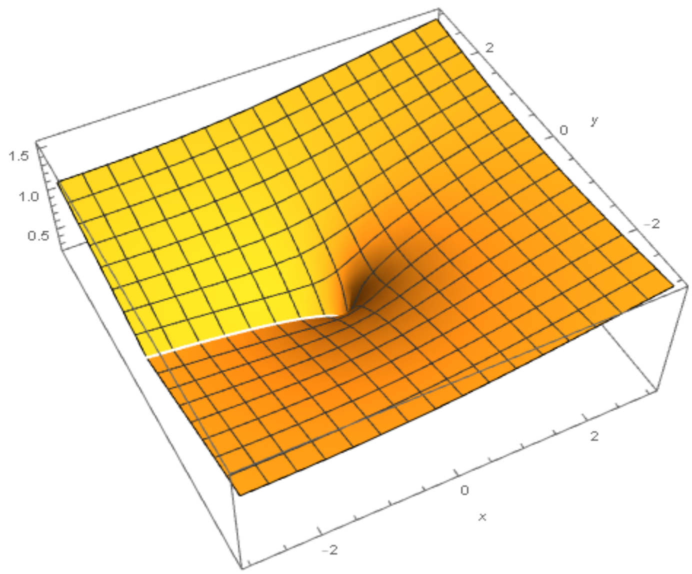
```
In[1]:= f[z_] := Exp[Log[z] / 3]

In[2]:= u[z_] := Re[f[z]]

In[3]:= v[z_] := Im[f[z]]

In[4]:= Plot3D[u[x + I * y], {x, -3, 3}, {y, -3, 3},
          AxesLabel → Automatic]
```

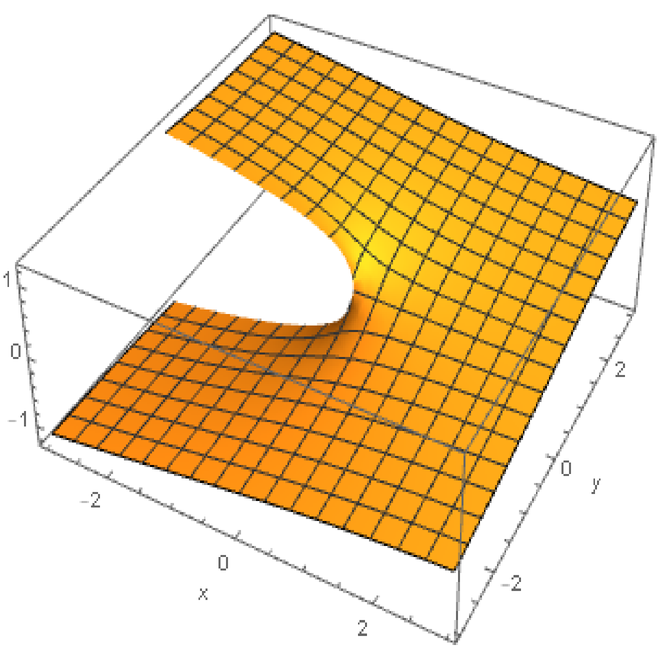
Out[4]=



(i) Graph of $u(z)$

```
In[5]:= Plot3D[v[x + I * y], {x, -3, 3}, {y, -3, 3},
          AxesLabel → Automatic]
```

Out[5]=



(ii) Graph of $v(z)$

□

(b) Determine the region of continuity of f.

Proof. We can see from the graph in the above that $u(z)$ is continuous everywhere except at the origin $(0,0)$, and $v(z)$ is discontinuous on the line $x \leq 0$, $y = 0$. Therefore, we conclude that f is continuous on the region:

$$\mathbb{C} \setminus \{z = x + iy \,:\, x \leq 0, y = 0\}$$

□

Exercise 4.

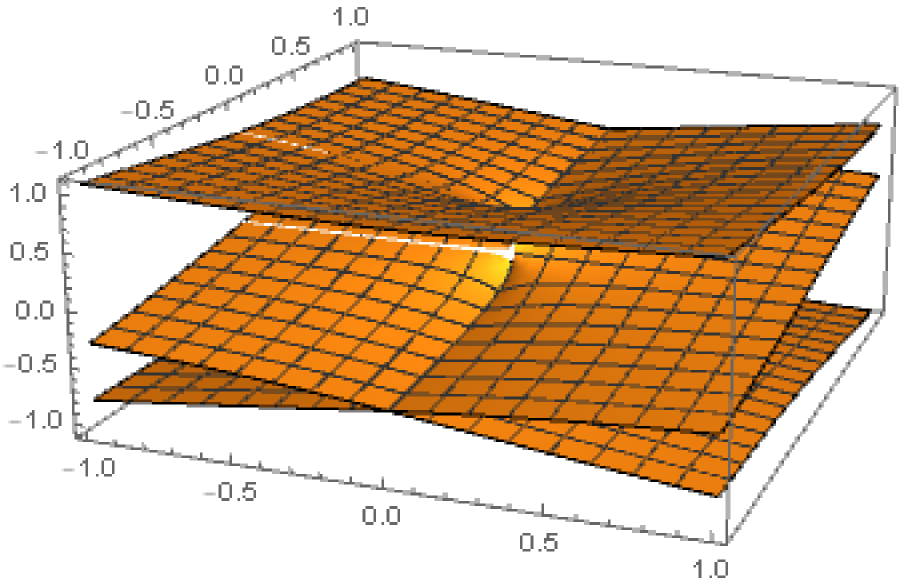
$$g(z) = \sqrt[3]{z} = e^{\frac{1}{3}\log z} = e^{\frac{1}{3}(\mathrm{Log} z + k2\pi i)}$$

where $k = 0, 1, 2$.

(a)

```
p[k_] := Plot3D[
  Re[Exp[1/3*(Log[x + I*y] + 2*k*Pi*I)]], {x, -1, 1}, {y, -1, 1}]
Show[p[0], p[1], p[2], PlotRange -> All]
```

Note that graph $p[3]$ is the same as $p[0]$, graph $p[4]$ is the same as $p[1]$, etc. This is because the exponential function is periodic with period $2\pi i$. Therefore, we only need to show $p[0]$, $p[1]$, $p[2]$.

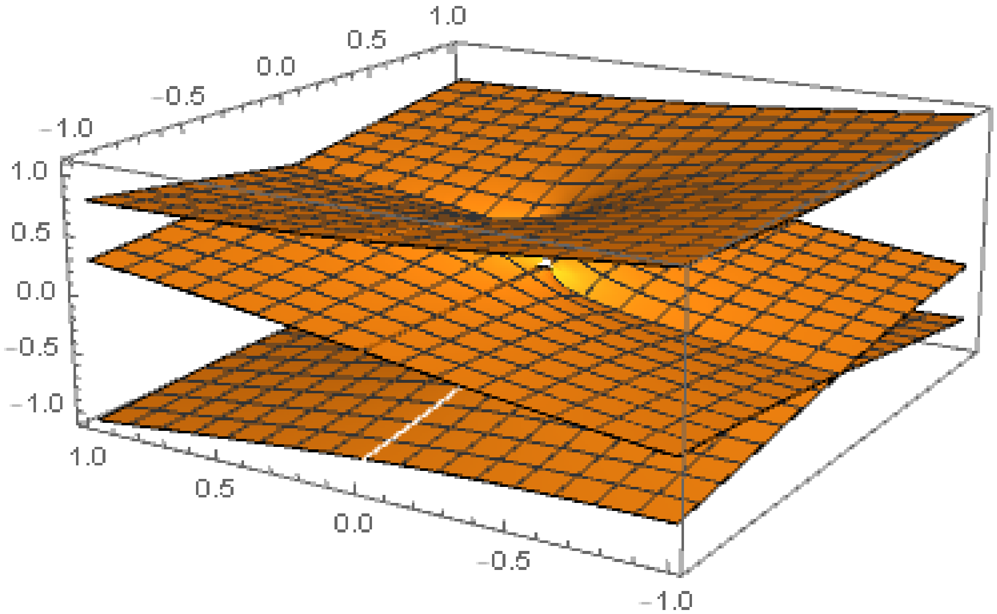


Graph of the real part of $\sqrt[3]{z}$

(b)

```
p[k_] := Plot3D[
  Im[Exp[1/3*(Log[x + I*y] + 2*k*Pi*I)]], {x, -1, 1}, {y, -1, 1}]
Show[p[0], p[1], p[2], PlotRange -> All]
```

Note that graph $p[3]$ is the same as $p[0]$, graph $p[4]$ is the same as $p[1]$, etc. This is because the exponential function is periodic with period $2\pi i$. Therefore, we only need to show $p[0]$, $p[1]$, $p[2]$.



Graph of the imaginary part of $\sqrt[3]{z}$