MTH 483/583 Complex Variables – Homework 3

Solution Key

Spring 2020

Exercise 1. Write the following complex numbers in either standard or polar form.

(a) $(1+i\sqrt{3})^{i+1}$

Answer: Since

$$1+i\sqrt{3} = 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 2e^{i(\frac{\pi}{3}+k2\pi)}$$

Thus,

$$(1+i\sqrt{3})^{i+1} = e^{(i+1)\log(1+i\sqrt{3})} = e^{(1+i)(\ln 2+i(\frac{\pi}{3}+k2\pi))}$$

$$= \boxed{e^{(\ln 2-\frac{\pi}{3}-k2\pi)+i(\ln 2+\frac{\pi}{3}+k2\pi)}}, \quad k\in\mathbb{Z}$$

□

(b) $(-1)^{\sqrt{2}}$

Answer:

$$(-1)^{\sqrt{2}} = e^{\sqrt{2}\log(-1)} = e^{\sqrt{2}i(-\pi+k2\pi)}$$

$$= \boxed{e^{i(-\sqrt{2}+2\sqrt{2}k)\pi}}, \quad k\in\mathbb{Z}$$

□

(c) $(-1-i)^{1/4}$

Answer:

$$(-1-i)^{1/4} = e^{\frac{1}{4}\log(-1-i)} = e^{\frac{1}{4}(\ln\sqrt{2}+i(-\frac{3\pi}{4}+k2\pi))}$$

$$= \boxed{2^{\frac{1}{8}}\left(\cos(-\frac{3\pi}{16}+\frac{k\pi}{2})+i\sin(-\frac{3\pi}{16}+\frac{k\pi}{2})\right)}, \quad k\in\mathbb{Z}$$

□

(d) $\log\left(e^{i\frac{19}{4}\pi}\right)$

Answer:

$$\log\left(e^{i\frac{19}{4}\pi}\right) = \log\left(e^{i(\frac{19}{4}\pi + k2\pi)}\right)$$

$$= \boxed{i(\frac{19}{4}\pi + k2\pi)}, \quad k \in \mathbb{Z}$$

□

(e) $e^{i\sqrt{2}}$

Answer: Note that $e^{i\sqrt{2}}$ is not computed as $e^{i\sqrt{2}\log e}$. We know that

$$e^{i\sqrt{2}} = \boxed{\cos(\sqrt{2}) + i\sin(\sqrt{2})}$$

□

Exercise 2. Find all complex numbers satisfying the following equation. Write your results in either standard form or polar form.

(a) $e^z = -1$.

Answer: Write $z = a + ib$, then

$$e^z = e^{a+ib} = e^a e^{ib} = -1 = e^{i(2k+1)\pi}.$$

Therefore,

$$\boxed{z = i(2k+1)\pi}$$

Alternatively, one can directly write $z = \log(-1) = \ln|-1| + iArg(-1) = i(\pi + 2k\pi)$. □

(b) $2^z = 4$.

Proof. There are two ways to solve this problem.
Method 1: Because $2^z = e^{z\log 2}$, we get $e^{z\log 2} = 4$. This equation simply means $z\log 2 = \log 4$. Thus,

$$\begin{aligned} z &= \frac{\log 4}{\log 2} = \frac{\ln 4 + ik2\pi}{\ln 2 + il2\pi} = \frac{(\ln 4 + ik2\pi)(\ln 2 - il2\pi)}{(\ln 2 + il2\pi)(\ln 2 - il2\pi)} \\ &= \frac{(\ln 4)(\ln 2) + kl4\pi + i2\pi(k\ln 2 - l\ln 4)}{(\ln 2)^2 + (l2\pi)^2} \\ &= \frac{(\ln 4)(\ln 2) + kl4\pi}{(\ln 2)^2 + (l2\pi)^2} + i\frac{2\pi(k\ln 2 - l\ln 4)}{(\ln 2)^2 + (l2\pi)^2}. \end{aligned}$$

Method 2: Write $z = a + ib$, then

$$2^z = 2^{a+ib} = e^{(a+ib)(\log 2)} = e^{(a+ib)(\ln 2 + ik2\pi)}$$

$$= e^{(a\ln 2 - bk2\pi)}\, e^{i(b\ln 2 + ak2\pi)}$$

Thus, $2^z = 4$ is equivalent to

$$e^{(a\ln 2 - bk2\pi)}\, e^{i(b\ln 2 + ak2\pi)} = 4 = 4\, e^{i2m\pi}, \quad m \in \mathbb{Z}$$

So,

$$\begin{cases} a\ln 2 - bk2\pi = 2\ln 2 \\ b\ln 2 + ak2\pi = 2m\pi, \quad m \in \mathbb{Z} \end{cases} \tag{1}$$

This is a linear system of two unknowns (a and b). Solving this systems of equations, we obtain

$$a = \left(\ln 4 + \frac{4km\pi^2}{\ln 2}\right) / \left(\ln 2 - \frac{4k^2\pi^2}{\ln 2}\right)$$

and

$$b = \frac{1}{\ln 2}\left[2m\pi - 2k\pi\left(\ln 4 + \frac{4km\pi^2}{\ln 2}\right) / \left(\ln 2 - \frac{4k^2\pi^2}{\ln 2}\right)\right]$$

for $m, k \in \mathbb{Z}$. □

(c) $\sin z = 1 - 2i$

$z = \arcsin(1-2i) = \frac{1}{i}\log(i(1-2i+\sqrt{1-(1-2i)^2})) = -i\log(2+i+\sqrt{4+4i})$. We have

$$\sqrt{4+4i} = \sqrt{\sqrt{32}e^{i\frac{\pi}{4}}} = \pm\sqrt[4]{32}e^{i\frac{\pi}{8}}.$$

Thus,

$$z = -i\log(2+i\pm\sqrt[4]{32}e^{i\frac{\pi}{8}})$$

If the plus sign is taken then

$$\begin{aligned} z &= -i\log(2+i+\sqrt[4]{32}e^{i\frac{\pi}{8}}) = -i\log\left(2+\sqrt[4]{32}\cos\frac{\pi}{8} + i\left(1+\sqrt[4]{32}\sin\frac{\pi}{8}\right)\right) \\ &= -i(\ln r + i(\theta + k2\pi)) \\ &= \theta + k2\pi - i\ln r \end{aligned}$$

where $r \approx 4.6116$ and $\theta \approx 0.4271$. Thus,

$$z \approx 0.4271 + k2\pi - i1.5286$$

If the minus sign is taken then

$$\begin{aligned} z &= -i\log(2+i-\sqrt[4]{32}e^{i\frac{\pi}{8}}) = -i\log\left(2-\sqrt[4]{32}\cos\frac{\pi}{8} + i\left(1-\sqrt[4]{32}\sin\frac{\pi}{8}\right)\right) \\ &= -i(\ln s + i(\gamma + m2\pi)) \\ &= \gamma + k2\pi - i\ln s \end{aligned}$$

where $s \approx 0.2168$ and $\gamma \approx 2.7145$. Thus,

$$z \approx -1.52857 + m2\pi - i2.7145$$

In conclusion, the solutions of the equation $\sin z = 1 - 2i$ are

$$z \in \{0.4271 + k2\pi - i1.5286 : \; k \in \mathbb{Z}\} \cup \{-1.52857 + m2\pi - i2.7145 : \; m \in \mathbb{Z}\}.$$

(d) $\cos z = 1/2$

Answer: Note that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{1}{2} \Leftrightarrow e^{iz} + e^{-iz} = 1$$

$$\Leftrightarrow e^{2i\pi} - e^{i\pi} + 1 = 0$$

Again, solving this equation by quadratic formula,

$$e^{iz} = \frac{1 \pm \sqrt{3}}{2}$$

We get $iz = \log\left(\frac{1 \pm \sqrt{3}}{2}\right)$, or equivalently

$$z = -i \log\left(\frac{1 \pm i\sqrt{3}}{2}\right)$$

If the plus sign is taken then

$$z = -i \log\left(\frac{1 + i\sqrt{3}}{2}\right) = -i \log\left(e^{i\frac{\pi}{3}}\right) = -ii\left(\frac{\pi}{3} + 2k\pi\right) = \frac{\pi}{3} + 2k\pi$$

If the minus sign is taken then

$$z = -i \log\left(\frac{1 - i\sqrt{3}}{2}\right) = -i \log\left(e^{i\frac{-\pi}{3}}\right) = -ii\left(-\frac{\pi}{3} + 2m\pi\right) = -\frac{\pi}{3} + 2m\pi$$

In conclusion, the set all solutions z is

$$z \in \left\{\pm\frac{\pi}{3} + 2k\pi : \quad k \in \mathbb{Z}\right\}$$

□

Exercise 3. Decide if each of the following statements is true or false. If it is true, prove it. If it is false, give a counterexample.

(a) $Log(z) + Log(w) = Log(zw)$, $\forall z, w \in \mathbb{C}$, $z, w \neq 0$.

Proof. **False.**

Let $z = w = -1$. Then $Log(z) = Log(w) = \ln(1) + i\pi = i\pi$

so

$$Log(z) + Log(w) = i2\pi$$

but

$$Log(zw) = Log(1) = \ln(1) + 0i = 0$$

□

(b) $e^{Log(z)} = z$, $\forall z \in \mathbb{C}$, $z \neq 0$.

Proof. **True.**

Write $z = re^{i\theta}$, for $\theta \in (-\pi, \pi]$ and $r \neq 0$.
Then

$$Log(z) = \ln(r) + i\theta$$

so that

$$e^{Log(z)} = e^{\ln(r)+i\theta} = re^{i\theta} = z.$$

□

(c) $\sqrt{z^2 - 1} = \sqrt{z-1}\sqrt{z+1}$, $\forall z \in \mathbb{C}$, $z \neq \pm 1$.

Proof. **False.**

Let $z = -2$. Then

$$\sqrt{z^2 - 1} = \underbrace{\sqrt{3}}_{\text{real square root}}$$

But

$$\sqrt{z-1} = \sqrt{-3} = i\sqrt{3}$$

and

$$\sqrt{z+1} = \sqrt{-1} = i$$

so

$$\sqrt{z-1}\sqrt{z+1} = i^2\sqrt{3} = -\sqrt{3}$$

Therefore,

$$\sqrt{z^2 - 1} \neq \sqrt{z-1}\sqrt{z+1}$$

□

Exercise 4. Determine all $z \in \mathbb{C}$ such that $z^2 + 1 \in \mathbb{R}_{\leq 0}$. Here $\mathbb{R}_{\leq 0}$ denotes the negative real line.

Proof. Write $z = x + iy$. Then

$$z^2 + 1 = (x^2 - y^2 + 1) + i(2xy)$$

If $z^2 + 1 \in \mathbb{R}_{\leq 0}$, then

$$x^2 - y^2 + 1 \leq 0 \quad (2)$$
$$2xy = 0 \quad (3)$$

- $x = 0 \Longrightarrow -y^2 + 1 \leq 0 \Longrightarrow y \geq 1$ or $y \leq -1$
- $y = 0 \Longrightarrow x^2 + 1 \leq 0$ a contradiction!

Therefore,

$$z \in \{z = iy \,|\, y \in \mathbb{R} \,\text{and}\, |y| \geq 1\}$$

□

Note: Geometrically, if we write $z = re^{i\theta}$ in polar form, then $z^2 = r^2e^{i(2\theta)}$ and so the module has squared and the angle has doubled. In light of this, if $z^2 + 1 \in \mathbb{R}_{\leq 0}$, then the angle of $z^2 + 1$ is twice of the angle of z since add 1 to z^2 only shits z^2 horizontally and is nothing to do with the angle. Therefore,

$$2\theta = (2k + 1)\pi, \quad k \in \mathbb{Z}$$

or

$$\theta = (k + \frac{1}{2})\pi, \quad k \in \mathbb{Z}$$

Now, $Re(z^2 + 1) = r^2 \cos((2k + 1)\pi) + 1 \leq 0$ implies $-r^2 + 1 \leq 0$ or

$$r^2 \geq 1$$

So

$$z = ir, \quad r \geq 1.$$

Exercise 5. Before doing the following problem, please take a look at the supplemental material called "**Mapping properties of the exponential function (continued)**" posted on Canvas and course website. *Make sure to include the Mathematica codes you use and some brief comments.*

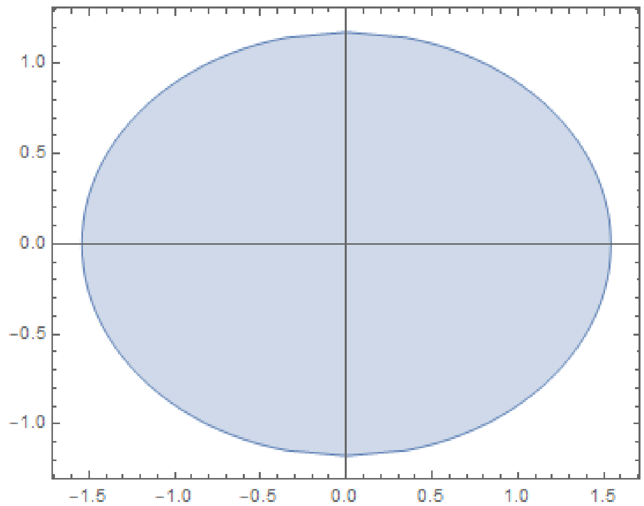
(a) Draw the image of the rectangle $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-1, 1]$ under $f(z) = \sin z$.

Answer:

```
In[1]:= f[z_] := Sin[z]

In[2]:= ParametricPlot[ReIm[f[t + s * I]], {t, -Pi * 0.5, Pi * 0.5}, {s, -1, 1}]
```

Out[2]=



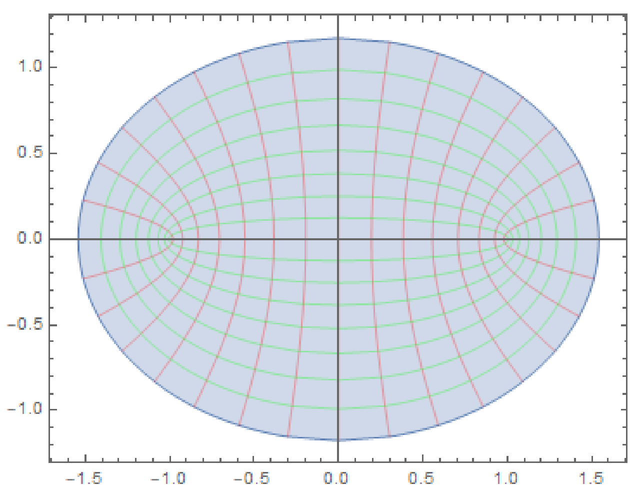
□

(b) Draw the images of some horizontal and vertical lines under f. Based on the picture, can you tell what the images of the line $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ are ?

Proof.

```
In[8]:= ParametricPlot[ReIm[f[t + s * I]], {t, -Pi * 0.5, Pi * 0.5}, {s, -1, 1},
         PlotRange → Full, Mesh → Automatic, MeshStyle → {Red, Green}]
```

Out[8]=



The function f maps the line $x = -\frac{\pi}{2}$ to the horizontal half-line: $x \leq -1$, and maps the line $x = \frac{\pi}{2}$ to the half-line: $x \geq 1$.

□

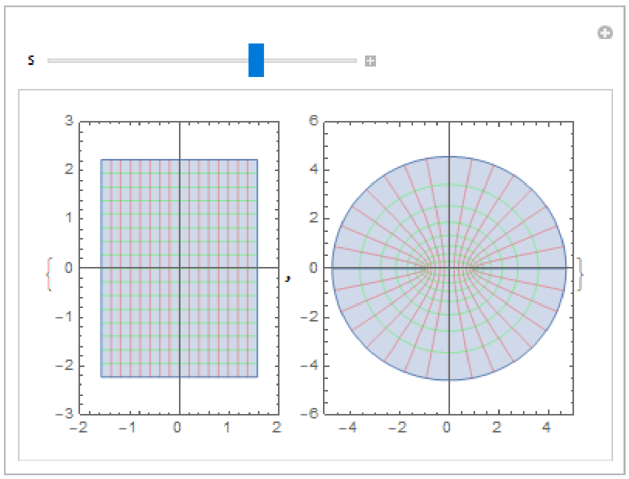
(c) f maps the vertical strip $[-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\infty, \infty)$ onto some region Ω. What is Ω? Is the map $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\infty, \infty) \longrightarrow \Omega$ a one-to-one mapping?

Proof. f maps the vertical strip $[-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\infty, \infty)$ one-to-one and onto the whole complex plane $\mathbb{C}$.

```
In[10]:= p[s_] := ParametricPlot[ReIm[x + y * I], {x, -Pi * 0.5, Pi * 0.5}, {y, -s, s},
           PlotRange → {{-2, 2}, {-3, 3}}, MeshStyle → {Red, Green}, Mesh → Automatic]

In[17]:= q[s_] := ParametricPlot[ReIm[f[x + y * I]], {x, -Pi * 0.5, Pi * 0.5}, {y, -s, s},
           PlotRange → {{-5, 5}, {-6, 6}}, MeshStyle → {Red, Green},
           Mesh → Automatic]
         Manipulate[{p[s], q[s]}, {s, 0.5, 3}]
```

Out[18]=



∎

(d) f maps the vertical strip $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\infty, \infty)$ onto some region D. What is D? Is the map $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\infty, \infty) \longrightarrow D$ a one-to-one mapping?

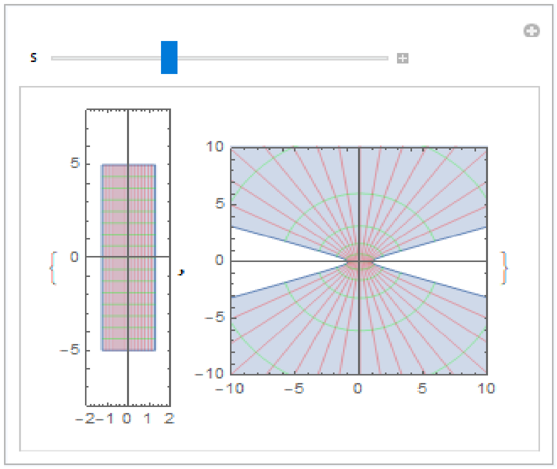
Proof. f maps the vertical strip $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\infty, \infty)$ one-to-one and onto the region: $\mathbb{C} \setminus (\mathbb{R}_{\leq -1} \cup \mathbb{R}_{\geq 1})$

```
In[32]:= p[s_] := ParametricPlot[ReIm[x + y * I], {x, -s, s}, {y, -5, 5},
           PlotRange → {{-2, 2}, {-8, 8}}, MeshStyle → {Red, Green}, Mesh → Automatic]

In[33]:= q[s_] := ParametricPlot[ReIm[f[x + y * I]], {x, -s, s}, {y, -5, 5},
           PlotRange → {{-10, 10}, {-10, 10}}, MeshStyle → {Red, Green},
           Mesh → Automatic]

In[34]:= Manipulate[{p[s], q[s]}, {s, Pi * 0.5 - 1, Pi * 0.5 + 1}]
```

Out[34]=

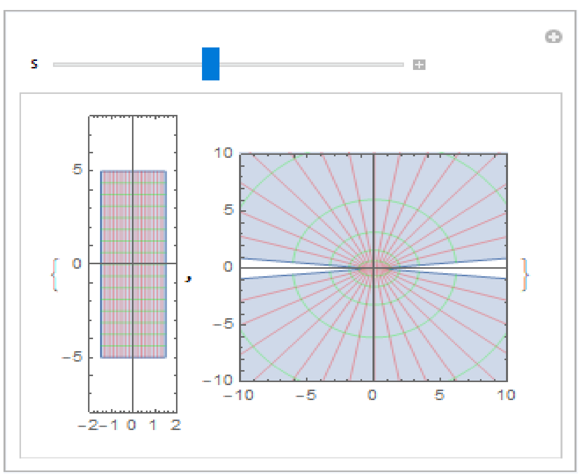



```
In[32]:= p[s_] := ParametricPlot[ReIm[x + y * I], {x, -s, s}, {y, -5, 5},
           PlotRange → {{-2, 2}, {-8, 8}}, MeshStyle → {Red, Green}, Mesh → Automatic]

In[33]:= q[s_] := ParametricPlot[ReIm[f[x + y * I]], {x, -s, s}, {y, -5, 5},
           PlotRange → {{-10, 10}, {-10, 10}}, MeshStyle → {Red, Green},
           Mesh → Automatic]

In[34]:= Manipulate[{p[s], q[s]}, {s, Pi * 0.5 - 1, Pi * 0.5 + 1}]
```


Out[34]=



```
In[32]:= p[s_] := ParametricPlot[ReIm[x + y * I], {x, -s, s}, {y, -5, 5},
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           Mesh → Automatic]

In[34]:= Manipulate[{p[s], q[s]}, {s, Pi * 0.5 - 1, Pi * 0.5 + 1}]
```
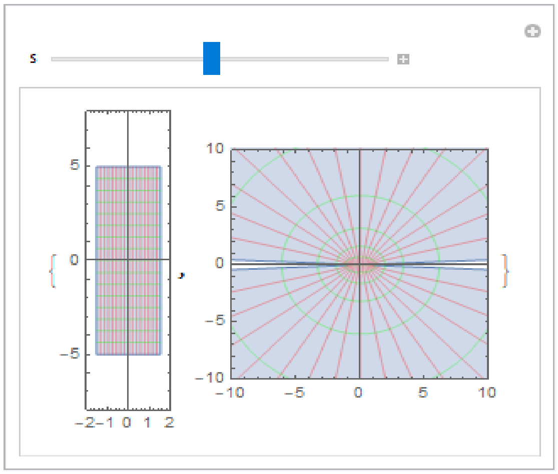

Out[34]=



□