MTH 483/583 Complex Variables – Homework 1

Solution Key

Spring 2020

Exercise 1. Write the following complex numbers in standard form $a+ib$.

(a) $(1+2i)(2-i)+(1+i)^2 = 2-i+4i-(2i)(i)+(1+2i+i^2) = \boxed{4+5i}$.

(b) $(2-i)^4 = (2-i)^2(2-i)^2 = (3-4i)(3-4i) = \boxed{-7-24i}$.

(c) $\frac{2+4i}{1+i}+(1+2i)^2 i^5 = \frac{(2+4i)(1-i)}{(1+i)(1-i)}+(-3+4i)i = \frac{6+2i}{2}+(-3i-4) = \boxed{-1-2i}$.

(d) $\frac{|-1-2i|}{3+i\sqrt{2}} = \frac{\sqrt{5}(3-i\sqrt{2})}{(3+i\sqrt{2})(3-i\sqrt{2})} = \boxed{\dfrac{3\sqrt{5}}{11}-\dfrac{\sqrt{10}}{11}i}$.

Exercise 2. Write the following complex numbers in polar form $re^{i\theta}$.

We use conventional principle argument $[-\pi,\pi)$ in this question.

(a) $\frac{1}{1+i}+\frac{1}{-1+i} = \frac{(-1+i)+(1+i)}{(1+i)(-1+i)} = -i = \boxed{e^{-i\frac{\pi}{2}}}$.

(b) $-2+i\sqrt{12} = 4\left(\frac{-2}{4}+i\frac{\sqrt{12}}{4}\right) = 4e^{i\theta}$, where $\theta = \pi+\arctan(-\frac{\sqrt{12}}{2}) = \pi-\frac{\pi}{3} = \frac{2\pi}{3}$

(c) Third roots of $1+i\sqrt{3}$:

Step 1: $1+i\sqrt{3} = 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 2e^{i\pi/3}$,
Step 2: $\sqrt[3]{1+i\sqrt{3}} = 2^{1/3}e^{i(\pi/9+2k\pi/3)}, \quad k=0,1,2$
Answer: $\boxed{2^{\frac{1}{3}}e^{i(\frac{\pi}{9}+\frac{2k\pi}{3})}}, \quad k=0,1,2$

(d) Fourth roots of $\frac{1}{2}-\frac{\sqrt{3}}{2}i$:

Step 1: $\frac{1}{2}-\frac{\sqrt{3}}{2}i = e^{-i\frac{\pi}{3}}$
Step 2: $\sqrt[4]{\frac{1}{2}-\frac{\sqrt{3}}{2}i} = e^{i(-\frac{\pi}{12}+\frac{k\pi}{2})}, \quad k=0,1,2,3$
Answer: $\boxed{e^{i(-\frac{\pi}{12}+\frac{k\pi}{2})}}, \quad k=0,1,2,3$

Exercise 3. Find all $z \in \mathbb{C}$ that satisfy the following equation.

(a) $|z| - 2z = i$:

Set $z = a + ib$:

$\Rightarrow \sqrt{a^2+b^2} - (2a + 2ib) = i$ or $(\sqrt{a^2+b^2} - 2a) - i(2b) = i$
so,

$\sqrt{a^2+b^2} - 2a = 0 \quad \text{and} \quad -2b = 1$

$\Rightarrow a = \frac{1}{\sqrt{12}}$, $b = \frac{-1}{2}$
$\therefore \boxed{z = \frac{1}{\sqrt{12}} - \frac{i}{2}}$

(b) $z^2\bar{z} = z$:

Since $z^2\bar{z} = z(z\bar{z}) = z|z|^2$ and $|z|^2 \in \mathbb{R} \implies |z|^2 z = z$,

so either $z = 0$ or $|z| = 1$. In conclusion, the set of solutions consists of 0 and all the complex numbers on the unit circle.

(c) $z^2 + (1+2i)z + i - 7 = 0$:

Use **quadratic formula**:

$z = \frac{-(1+2i) \pm \sqrt{(1+2i)^2 - 4(i-7)}}{2}$

$\therefore z = \boxed{-3 - i}, \boxed{2 - i}$

(d) $z^6 + z^3 + 1 = 0$:

Step 1: Let $u = z^3$. Then use quadratic formula on the equation $u^2 + u + 1 = 0$, we obtain

$$u = \frac{-1 \pm \sqrt{3}i}{2}$$

or

$$u = e^{\frac{2i\pi}{3}}, u = e^{\frac{-2i\pi}{3}}$$

Step 2: solve the equation $z^3 = e^{\frac{2i\pi}{3}}$:

$z = \boxed{e^{i(\frac{2\pi}{9} + \frac{2k\pi}{3})}}$, $k = 0, 1, 2$

Step 3: Similarly, solve the equation $z^3 = e^{\frac{-2i\pi}{3}}$:

$z = \boxed{e^{i(\frac{-2\pi}{9}+\frac{2k\pi}{3})}}$, $k = 0, 1, 2$

(e) $z^3 + (2-i)z^2 + (2-2i)z - 2i = 0$:

We know from the hint that $z = i$ is one of the roots. Factor the equation using long division:

$$0 = z^3 + (2-i)z^2 + (2-2i)z - 2i = (z-i)(z^2+2z+2)$$

and

$$z^2 + 2z + 2 = 0 \longrightarrow z = -1 \pm i$$

Therefore, $z = \boxed{i}, \boxed{-1-i}, \boxed{-1+i}$.

Exercise 4. Sketch the following sets on the complex plane. You can either sketch by hand or use Mathematica to plot.

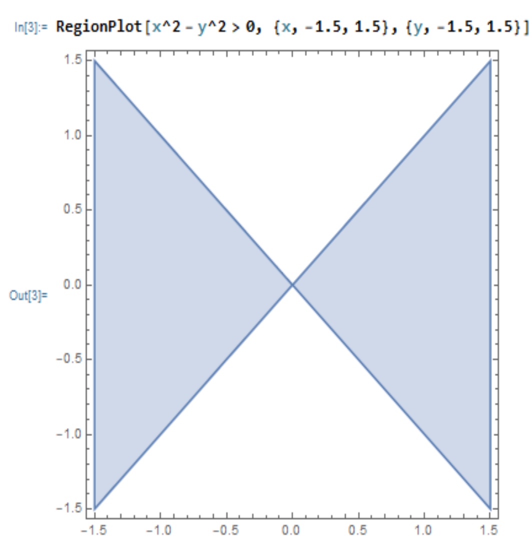
We set $z = x + iy$ in the following.

(a) $Re(z^2) > 0$:

Note that $Re(z^2) = x^2 - y^2$, therefore we want the region:

$$x^2 - y^2 > 0$$

in the complex plane.



(b) $0 < |z-1| < 2$:

$$0 < |z-1| = \sqrt{(x-1)^2 + y^2} < 2$$

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In[11]:= RegionPlot[0 < (x - 1)^2 + y^2 < 4, {x, -1.5, 3.5}, {y, -2.5, 2.5}]
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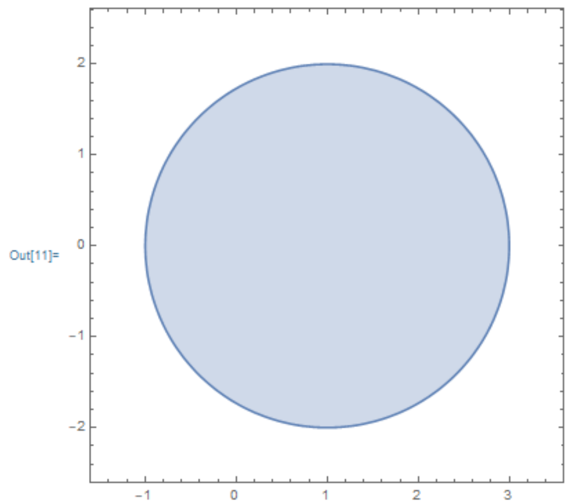

(c) $|z| < |z-4|$:

Notice that this is equivalent to $|z-0| < |z-4|$, namely, all the complex numbers z such that the distance between z and the origin 0 is strictly less than the distance between z and 4, which consists of all the points on the left of the vertical line $x = 2$:

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In[12]:= RegionPlot[Sqrt[x^2 + y^2] ≤ Sqrt[(x - 4)^2 + y^2], {x, -10, 10}, {y, -10, 10}]
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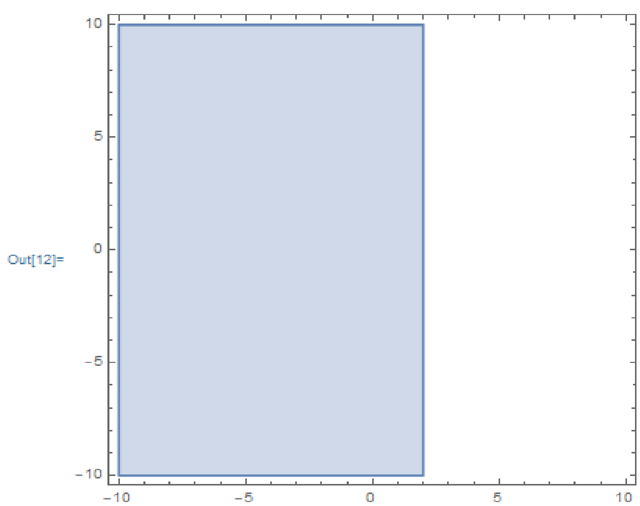
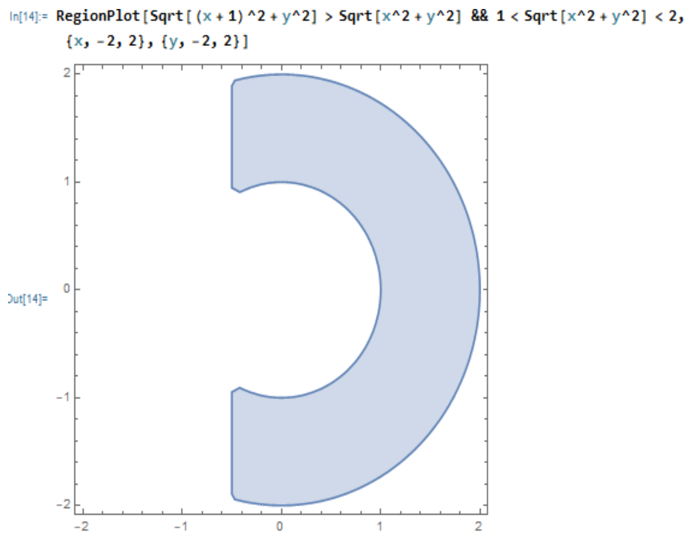


(d) $|z+1| > |z|$ and $1 < |z| < 2$:



(e) $Re(z) > Im(z)$ and $|z-1| < 1$:

This is the intersection of the half-plane $x > y$ (or the region below the line $y = x$) and the region inside the unit circle centered at $(1, 0)$: