Worksheet

11/27/2019

1. Let P_2 be the vector space of all polynomials of degree ≤ 2 with real coefficients. Let $F : P_2 \to P_2$ be a linear map given by $F(u) = xu'$. Is F diagonalizable? If so, diagonalize F (i.e. find a basis of P_2 in which F is represented by a diagonal matrix.)

First, we find all eigenvalues.

For each $u \in P_2$, we write $u = ax^2 + bx + c$.

We solve for all $u \in P_2$, $\lambda \in \mathbb{R}$ from the equation

$$F(u) = \lambda u. \qquad (*)$$

$$LHS(*) = xu' = x(2ax + b) = 2ax^2 + bx.$$

$$RHS(*) = \lambda a x^2 + \lambda b x + \lambda c.$$

For LHS = RHS, we need

$$\begin{cases} 2a = \lambda a \\ b = \lambda b \\ 0 = \lambda c \end{cases} \quad \text{or equivalently} \quad \begin{cases} (2-\lambda)a = 0, \\ (1-\lambda)b = 0, \\ \lambda c = 0. \end{cases}$$

If $\lambda \neq 0, 1, 2$ then $a = b = c = 0$. In this case $u = 0$.

Thus, λ is <u>not</u> an eigenvalue of F.

If $\lambda = 0$ then $a = b = 0$ and c is arbitrary in $\mathbb{R}$. Thus,

$$E(0) = \{u \in P_2 : F(u) = 0u\} = \{u = c \text{ (constant function)} : c \in \mathbb{R}\}$$

$E(0)$ is 1-dimensional with basis $B_1 = \{1\}$.

constant function 1

If $\lambda = 1$ then $a = c = 0$ and b is arbitrary in $\mathbb{R}$.

$$E(1) = \{u \in P_2 : F(u) = 1u\} = \{u = bx : b \in \mathbb{R}\}$$

Thus, $E(1)$ is 1-dim with basis $B_2 = \{x\}$.

Similarly, $E(2)$ is 1-dim with basis $B_3 = \{x^2\}$.
Because

$$\dim P_2 = 3 = \dim E(0) + \dim E(1) + \dim E(2),$$

F is diagonalizable. The basis that diagonalizes F is

$$B = B_1 \sqcup B_2 \sqcup B_3 = \{1, x, x^2\}.$$

The matrix that represents F in basis B is

$$[F]_B = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$