Worksheet

11/13/2019

1. Consider the following subspaces of $\mathbb{R}^4$:

$$\begin{aligned} V_1 &= \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 :\ x_1 = x_2 = 0\}, \\ V_2 &= \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 :\ x_1 = x_3 = 0\}, \\ V_3 &= \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 :\ x_2 = x_3 = x_4 = 0\}, \\ V_4 &= \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 :\ x_1 = x_3 = x_4 = 0\}. \end{aligned}$$

Which of the following sums are direct sums: (a) $V_1 + V_2$; (b) $V_2 + V_3 + V_4$; (c) $V_1 + V_3 + V_4$?

A basis of V_1 is $\mathcal{B}_1 = \{(0,0,1,0), (0,0,0,1)\}$.

" V_2 " $\mathcal{B}_2 = \{(0,1,0,0), (0,0,0,1)\}$.

" V_3 " $\mathcal{B}_3 = \{(1,0,0,0)\}$

" V_4 " $\mathcal{B}_4 = \{(0,1,0,0)\}$.

$V_1 + V_2$ is not a direct sum because $(0,0,0,1) \in V_1 \cap V_2$.

To see if $V_2 + V_3 + V_4$ is a direct sum, we concatenate $\mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$:

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{B}_2}\!\!\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathcal{B}_3}\!\!\underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathcal{B}_4} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

not pivot column

These 4 vectors are not linearly independent. Thus, $V_2 + V_3 + V_4$ is not a direct sum.

By a similar method, we can see that $V_1 + V_3 + V_4$ is a direct sum.

2. Let $F : P_3 \to P_3$ be a linear map given by $F(u) = xu'$. Consider the following subspaces $U = \text{span}\{x, 1\}$ and $V = \text{span}\{x^2 - 1, x + 1\}$. Check whether U and V are invariant under F.

* Check if U is invariant under F:

Let $u \in U$. We want to check if $F(u) \in U$.

By the def. of U, we can write $u = ax + b$ for some $a, b \in \mathbb{R}$.

Then $F(u) = xu' = ax$.

Then $F(u) \in U$. Thus, U is invariant under f.

* Check if V is invariant under F:

Let $v \in V$. We want to check if $F(v) \in V$.

By the definition of V, we can write
$$v = a(x^2 - 1) + b(x+1) = ax^2 + bx - a + b$$

Then
$$F(v) = xv' = x(2ax + b) = 2ax^2 + bx$$

For $a = 0$ and $b = 1$, we have $v = x + 1$ and $F(v) = x$.

We will show that $x \notin V = \text{span}\{x^2 - 1, x + 1\}$.

Suppose by contradiction that $x \in V$. Then there are $c, d \in \mathbb{R}$ such that

$$x = c(x^2 - 1) + d(x + 1). \qquad \forall x \in \mathbb{R}$$

Equivalently,
$$cx^2 + (d-1)x - c + d = 0 \qquad \forall x \in \mathbb{R}$$

This only happens if $\begin{cases} c = 0 \\ d - 1 = 0 \\ c + d = 0 \end{cases}$ However, this system is inconsistent.

Therefore, $F(v) = x \notin V$. We conclude that V is not invariant under F.