Worksheet

10/9/2019

1. View the determinant of 2×2 matrices as a map $\det\colon M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$. Is it a linear map? Verify your answer.

We want to show that det is <u>not</u> a linear map.

Let $A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\det A = 1$.

We have $2A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Then $\det 2A = 4$.

Because $\det 2A \neq 2 \det A$, det is not a linear map.

2. Consider the map $F : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ defined by $F(u)(x) = xu(x)$, written briefly as $F(u) = xu$. Show that F is a linear map.

We want to show that F is a linear map.

- Check if F is additive:

Let $u_1, u_2 \in \mathbb{R}^{\mathbb{R}}$. We want to show $F(u_1+u_2) = F(u_1) + F(u_2)$.

That is to show $F(u_1+u_2)(x) = (F(u_1)+F(u_2))(x)$ for all $x \in \mathbb{R}$.

Fix an $x \in \mathbb{R}$. By the definition of F,

$$LHS = x(u_1+u_2)(x) = x(u_1(x)+u_2(x)).$$

$$\begin{aligned} RHS &= F(u_1)(x) + F(u_2)(x) \\ &= xu_1(x) + xu_2(x) \\ &= x(u_1(x)+u_2(x)). \end{aligned}$$

We see that $LHS = RHS$. Thus, f is additive.

- Check if F is scalar multiplicative:

Let $c \in \mathbb{R}$ and $u \in \mathbb{R}^{\mathbb{R}}$. We want to show $F(cu) = cF(u)$.

That is to show $F(cu)(x) = (cF(u))(x)$ for all $x \in \mathbb{R}$.

Fix an $x \in \mathbb{R}$. By the definition of F,

$$LHS = x(cu)(x) = xcu(x).$$

$$RHS = cF(u)(x) = cxu(x).$$

We see that $LHS = RHS$. Hence, f is scalar multiplicative.