Worksheet

10/7/2019

1. Let V be the set of all polynomials $f(x)$ of degree ≤ 2, with real coefficients, such that $f(1) = 0$. Is V a vector space over $\mathbb{R}$? If so, find a basis of V and the dimension of V.

We showed in previous class that V is a vector space over $\mathbb{R}$.

A vetor in V is of the form $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$.

The condition $f(1) = 0$ is equivalent to $a+b+c=0$. This gives $c = -a-b$.

Then

$$V = \{ \text{functions } f(x) = ax^2 + bx + (-a-b) \text{ where } a, b \in \mathbb{R} \}$$

$$= \{ f(x) = a(x^2-1) + b(x-1) \text{ where } a, b \in \mathbb{R} \}$$

$$= \text{span}\{ \underbrace{x^2-1}_{y_1}, \underbrace{x-1}_{y_2} \}$$

V is the linear span of y_1 and y_2. To check if $\{y_1, y_2\}$ is a basis of V, we only need to check if y_1 and y_2 are linearly independent.

Consider the equation: $a y_1 + b y_2 = 0$. This is equivalent to

$$ax^2 + bx + (-a-b) = 0 \quad \text{for all } x \in \mathbb{R}.$$

This is a polynomial of degree ≤ 2. For it to have more than 2 roots, all coefficients must be equal to zero. This means

$$a = b = -a-b = 0$$

Thus, $a = b = 0$. We conclude that $\{y_1, y_2\}$ is a basis of V and

$$\dim_{\mathbb{R}} V = 2$$

2. Let V be the set of all 2×2 matrices with real coefficients such that the sum of each row is equal to 0. Is V a vector space over $\mathbb{R}$? If so, find a basis of V and the dimension of V.

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R},\ a+b = c+d = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : b = -a,\ d = -c \right\}$$

$$= \left\{ \begin{bmatrix} a & -a \\ c & -c \end{bmatrix} : a, c \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} a & -a \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & -c \end{bmatrix} : a, c \in \mathbb{R} \right\}$$

$$= \left\{ a \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} : a, c \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}}_{E_1}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}}_{E_2} \right\}$$

We see that V is a linear span of $\{E_1, E_2\}$.

To check if $\{E_1, E_2\}$ is a basis of V, we only need to check if it is a linearly independent set. Consider the equation

$$c_1 E_1 + c_2 E_2 = 0$$

This equation is equivalent to

$$c_1 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Because $LHS = \begin{bmatrix} c_1 & -c_1 \\ c_2 & -c_2 \end{bmatrix}$ and $RHS = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

$$c_1 = c_2 = 0.$$

Therefore, $\{E_1, E_2\}$ is a basis of V and $\dim_{\mathbb{R}} V = 2$.