Worksheet

10/28/2019

1. Consider a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x + y,\ x + 2y)$. Is f a monomorphism, epimorphism or isomorphism?

In this problem, $V = W = \mathbb{R}^2$. We'll check if f is monomorphic.

Let $v \in \mathbb{R}^2$ be such that $f(v) = 0$. We want to show $v = 0$. Write $v = (x, y)$.

We have $f(v) = f(x, y) = (x+y, x+2y)$.

Because $f(x,y) = (0,0)$, we have

$$\begin{cases} x + y = 0, \\ x + 2y = 0. \end{cases}$$

.... From these equations, we get $x = y = 0$. Thus, $v = (0, 0)$.

Conclusion: f is monomorphic. Since $\dim V = \dim W$, f is also isomorphic.

2. Consider a linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x, y, z) = (x + y,\ y + z,\ z + x)$. Is f a monomorphism, epimorphism or isomorphism?

In this problem, $V = W = \mathbb{R}^3$. We'll check if f is monomorphic.

Let $v \in \mathbb{R}^3$ be such that $f(v) = 0$. We want to show $v = 0$. Write $v = (x, y)$.

We have $f(v) = f(x, y) = (x+y, y+z, z+x)$.

Because $f(x,y) = (0,0,0)$, we have

$$\begin{cases} x + y = 0, \\ y + z = 0, \\ z + x = 0. \end{cases}$$

From the first and second equation, we get $x = -y$ and $z = -y$. Substitute x and z into the third equation:

$$-2y = 0.$$

Then $y = 0$, which implies $x = z = 0$. Thus, $v = (0, 0, 0)$.

Conclusion: f is monomorphic. Since $\dim V = \dim W$, f is also isomorphic.

3. Consider a linear map $F : P_3 \to P_2$ given by $F(u) = u'$. Here P_n denotes the vector space of all polynomials of real coefficients with degree $\leq n$. Is f a monomorphism, epimorphism or isomorphism?

In this problem, $V = P_3$ and $W = P_2$. Because

$$\dim V = 4 > \dim W = 2,$$

F is not monomorphic (thus, not isomorphic). We'll check if F is epimorphic.

$$\text{range}(F) = \{F(u) : u \in P_3\} = \{u' : u \in P_3\}$$

A vector $u \in P_3$ is of the form $u = ax^3 + bx^2 + cx + d$ where a, b, c, d are real constants. Then

$$u' = 3ax^2 + 2bx + c.$$

Thus,

$$\text{range}(F) = \{3ax^2 + 2bx + c : a, b, c \in \mathbb{R}\}$$
$$= \text{Span}\{x^2, x, 1\}$$

The set $\{x^2, x, 1\}$ is the standard basis of P_2. Thus, $\text{range}(F) = W$.

Conclusion: F is epimorphic, but not monomorphic.