Worksheet

10/23/2019

1. Consider a linear map $f : \mathbb{R}^4 \to M_{1\times 2}(\mathbb{R})$ given by

$$f(a,b,c,d) = \begin{bmatrix} a+b & a+b \end{bmatrix}$$

Is f a monomorphism, epimorphism or isomorphism?

In this problem, $V = \mathbb{R}^4$ and $W = M_{1\times 2}(\mathbb{R})$. We see that

$$\dim V = 4 > \dim W = 2.$$

Therefore, f is not a monomorphism. Consequently, it is not an isomorphism. To check if f is an epimorphism, we check if $\text{range}(f) = W$.

$$\begin{aligned}
\text{range}(f) &= \{ f(a,b,c,d) : a,b,c,d \in \mathbb{R} \} \\
&= \{ [a+b \quad a+b] : a,b,c,d \in \mathbb{R} \} \\
&= \{ [a+b \quad a+b] : a,b \in \mathbb{R} \} \\
&= \{ (a+b) [1 \quad 1] : a,b \in \mathbb{R} \} \\
&= \text{span} \{ [1 \quad 1] \}
\end{aligned}$$

This is a one-dimensional vector space. Since W is two-dimensional, $\text{range}(f)$ is strictly smaller than W. Therefore, f is not an epimorphism.

2. Consider a linear map $f : \mathbb{R}^4 \to M_{2\times 2}(\mathbb{R})$ given by

$$f(a,b,c,d) = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Is f a monomorphism, epimorphism or isomorphism?

In this problem, $V = \mathbb{R}^4$ and $W = M_{2\times 2}(\mathbb{R})$. We have

$\dim V = 4 = \dim W$.

We now check if f is monomorphic.

$$\mathrm{null}(f) = \{(a,b,c,d) : f(a,b,c,d) = 0\}$$

$$= \left\{(a,b,c,d) : \begin{bmatrix} b & a \\ d & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$$

$$= \{(a,b,c,d) : a = b = c = d = 0\}$$

$$= \{(0,0,0,0)\}.$$

Thus, f is a monomorphism.

Because $\dim V = \dim W$, f is also an epimorphism and an isomorphism.