Worksheet

10/16/2019

1. Let P_2 be the vector spaces of all polynomials of degree ≤ 2, with real coefficients. The set $B_0 = \{1, t, t^2\}$ is the standard basis of P_2. Consider another basis $B = \{2, 1-t, 2t^2+1\}$. Find the coordinate vector of the polynomial $f(t) = -2t^2 - 3t$ in basis B.

$$B_0 = \{\underbrace{1}_{e_1}, \underbrace{t}_{e_2}, \underbrace{t^2}_{e_3}\}$$

$$B = \{\underbrace{2}_{f_1}, \underbrace{1-t}_{f_2}, \underbrace{2t^2+1}_{f_3}\}$$

We want to find $[f]_B$, that is, a colum vector $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ such that $f = c_1 f_1 + c_2 f_2 + c_3 f_3$.

This equation can be rewritten as

$$-2t^2 - 3t = c_1 2 + c_2(1-t) + c_3(2t^2+1)$$

$$\text{RHS} = 2c_3 t^2 + (-c_2)t + (2c_1 + c_2 + c_3)$$

For LHS=RHS for all t, the coefficients of each power of t must match. Thus,

$$\begin{cases} 2c_3 = -2 \\ -c_2 = -3 \\ 2c_1 + c_2 + c_3 = 0 \end{cases}$$

This results in $\begin{cases} c_3 = -1 \\ c_2 = 3 \\ c_1 = -1 \end{cases}$

Therefore, $[f]_B = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$.

2. Consider a linear map $f : M_{2\times 2}(\mathbb{R}) \to M_{1\times 3}(\mathbb{R})$ given by

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b & c+d & a+b \end{bmatrix}$$

(a) Find a basis for null(f). What is its dimension?

(b) Find a basis for range(f). What is its dimension?

(a)

$$\text{null}(f) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = [0 \ 0 \ 0] \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : [a+b \ \ c+d \ \ a+b] = [0 \ 0 \ 0] \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a+b=0, \ c+d=0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : b=-a, \ d=-c \right\}$$

$$= \left\{ \begin{bmatrix} a & -a \\ c & -c \end{bmatrix} : a, c \in \mathbb{R} \right\}$$

$$= \left\{ a \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} : a, c \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}}_{A_2} \right\}$$

To check if $\{A_1, A_2\}$ is a basis of null(A), we need to check if it is linear independent. Consider $c_1, c_2 \in \mathbb{R}$ satisfying

$$c_1 A_1 + c_2 A_2 = 0.$$

This is equivalent to

$$c_1 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which is equivalent to

$$\begin{bmatrix} c_1 & -c_1 \\ c_2 & -c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which gives $c_1 = c_2 = 0$. Therefore, $\{A_1, A_2\}$ is linearly independent. It is a basis of null(A). The dimension of null(A) is 2.

(b) $\text{range}(f) = \left\{ f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) : a, b, c, d \in \mathbb{R} \right\}$

$$= \{ [a+b \quad c+d \quad a+b] : a,b,c,d \in \mathbb{R} \}$$

$$= \{ [a \quad c \quad a] + [b \quad d \quad b] : a,b,c,d \in \mathbb{R} \}$$

$$= \{ [a \quad 0 \quad a] + [0 \quad c \quad 0] + [b \quad 0 \quad b] + [0 \quad d \quad 0] : a,b,c,d \in \mathbb{R} \}$$

$$= \{ a[1 \quad 0 \quad 1] + c[0 \quad 1 \quad 0] + b[1 \quad 0 \quad 1] + d[0 \quad 1 \quad 0] : a,b,c,d \in \mathbb{R} \}$$

$$= \{ (a+b)[1 \quad 0 \quad 1] + (c+d)[0 \quad 1 \quad 0] : a,b,c,d \in \mathbb{R} \}$$

$$= \text{span}\{ \underbrace{[1 \quad 0 \quad 1]}_{D_1}, \underbrace{[0 \quad 1 \quad 0]}_{D_2} \}$$

To check if $\{D_1, D_2\}$ is linearly independent, we consider c_1 and c_2 satisfying $c_1 D_1 + c_2 D_2 = 0$. This eq. is equivalent to

$$c_1 [1 \quad 0 \quad 1] + c_2 [0 \quad 1 \quad 0] = [0 \quad 0 \quad 0]$$

which is equiv. to

$$[c_1 \quad c_2 \quad c_1] = [0 \quad 0 \quad 0]$$

which is equiv. to

$$\begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$

Therefore, $\{D_1, D_2\}$ is a basis of range(f). The dimension is 2.