MTH 342 Worksheet 9

Week 8 – 11/21/2019

Name: ______________________________ **Recitation time:** __________

1. Suppose $T\colon V \to V$ is a linear map, and let U_1 and U_2 be invariant subspaces of T.

(a) Prove that $U_1 + U_2$ is an invariant subspace of T.

Solution: Let $u_1 + u_2$ be an arbitrary element of $U_1 + U_2$ where $u_1 \in U_1$ and $u_2 \in U_2$. Since U_1 and U_2 are invariant subspaces of T, we have $T(u_1) \in U_1$ and $T(u_2) \in U_2$. Therefore

$$T(u_1 + u_2) = T(u_1) + T(u_2) \in U_1 + U_2$$

so $U_1 + U_2$ is an invariant subspace of T.

(b) Prove that $U_1 \cap U_2$ is an invariant subspace of T.

Solution: Let u be an arbitrary element of $U_1 \cap U_2$. Since U_1 and U_2 are invariant subspaces of T, we have $T(u) \in U_1$ and $T(u) \in U_2$, so $T(u) \in U_1 \cap U_2$. Therefore, $U_1 \cap U_2$ is an invariant subspace of T.

2. Let V_1 and V_2 be vector spaces over $\mathbb{C}$ defined by

$$V_1 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 - iz_2 = 0\},$$
$$V_2 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + (1+i)z_2 = (1-i)z_3\}.$$

(a) Find bases for V_1 and V_2.

Solution:

$$\begin{aligned} V_1 &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 - iz_2 = 0\} \\ &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 = iz_2\} \\ &= \{(iz_2, z_2, z_3) : z_2, z_3 \in \mathbb{C}\} \\ &= \{z_2(i, 1, 0) + z_3(0, 0, 1) : z_2, z_3 \in \mathbb{C}\} \end{aligned}$$

so $B_1 = \{(i, 1, 0), (0, 0, 1)\}$ is a basis for V_1.

$$\begin{aligned} V_2 &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + (1+i)z_2 = (1-i)z_3\} \\ &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 = -(1+i)z_2 + (1-i)z_3\} \\ &= \{(-(1+i)z_2 + (1-i)z_3, z_2, z_3) : z_2, z_3 \in \mathbb{C}\} \\ &= \{z_2(-1-i, 1, 0) + z_3(1-i, 0, 1) : z_2, z_3 \in \mathbb{C}\} \end{aligned}$$

so $B_2 = \{(-1-i, 1, 0), (1-i, 0, 1)\}$ is a basis for V_2.

(b) Is the sum $V_1 + V_2$ a direct sum?

Solution: No. Notice that $\dim(V_1) = \dim(V_2) = 2$, but $V_1 + V_2 \subseteq \mathbb{C}^3$ so $\dim(V_1 + V_2) \leq 3$. Therefore

$$\dim(V_1 + V_2) \neq \dim(V_1) + \dim(V_2),$$

so $V_1 + V_2$ is not a direct sum of V_1 and V_2.

3. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{bmatrix}.$$

The eigenvalues of A are $\lambda = 1, 2, 4$. Let

$$\begin{aligned} E_1 &= \{v \in \mathbb{R}^3 : Av = v\} \\ E_2 &= \{v \in \mathbb{R}^3 : Av = 2v\} \\ E_4 &= \{v \in \mathbb{R}^3 : Av = 4v\} \end{aligned}$$

(a) Find bases for E_1, E_2, and E_4.

Solution: Given any $\lambda \in \mathbb{R}$, the equation $Av = \lambda v$ can be rewritten as

$$\begin{aligned} Av &= \lambda v \\ &\downarrow \\ Av - \lambda v &= 0 \\ &\downarrow \\ (A - \lambda I)v &= 0 \end{aligned}$$

Where I is the identity matrix.

To find E_1, we solve the equation $(A - I)v = 0$ for $v = (v_1, v_2, v_3)$. That is,

$$\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Write this is equation in augmented form and use row reduction:

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 + R_1 \to R_3} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_3 \to R_3} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore $v_2 = v_3 = 0$ and v_1 is a free variable, so $E_1 = \{(v_1, 0, 0) : v_1 \in \mathbb{R}\}$ has basis $B_1 = \{(1, 0, 0)\}$.

(Continued on next page)

For E_2 and E_4 I will not write all of the row reduction details.

The augmented matrix for E_2 is

$$\left[\begin{array}{ccc|c} 1-2 & 1 & 0 & 0 \\ 0 & 2-2 & 0 & 0 \\ 0 & -1 & 4-2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore $v_1 = 2v_3$ and $v_2 = 2v_3$, where v_3 is a free variable. Thus $E_1 = \{(2v_3, 2v_3, v_3) : v_3 \in \mathbb{R}\}$ has basis $B_2 = \{(2, 2, 1)\}$.

The augmented matrix for E_4 is

$$\left[\begin{array}{ccc|c} 1-4 & 1 & 0 & 0 \\ 0 & 2-4 & 0 & 0 \\ 0 & -1 & 4-4 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore $v_1 = v_2 = 0$ and v_3 is a free variable, so $E_4 = \{(0, 0, v_3) : v_3 \in \mathbb{R}\}$ has basis $B_4 = \{(0, 0, 1)\}$.

(b) Is the sum $E_1 + E_2 + E_4$ a direct sum?
Solution: Yes. Let

$$B = B_1 \cup B_2 \cup B_3 = \{(1, 0, 0), (2, 2, 1), (0, 0, 1)\}$$

so $E_1 + E_2 + E_4 = \operatorname{span}(B)$. Then

$$(0, 1, 0) = \tfrac{1}{2}(2, 2, 1) - (1, 0, 0) - \tfrac{1}{2}(0, 0, 1)$$

and thus $E_1 + E_2 + E_4$ contains the standard basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ for $\mathbb{R}^3$. Therefore $E_1 + E_2 + E_4 = \mathbb{R}^3$. Calculating dimensions gives

$$\begin{aligned} \dim(E_1) + \dim(E_2) + \dim(E_4) &= 1 + 1 + 1 \\ &= 3 \\ &= \dim(\mathbb{R}^3) \end{aligned}$$

so $E_1 + E_2 + E_4$ is a direct sum of E_1, E_2, and E_4, and can be written as $E_1 \oplus E_2 \oplus E_4$.