MTH 342 Worksheet 8

Week 7 – 11/14/2019

Name: **Answer Key** **Recitation time:** ________

1. Let U, V, and W be subspaces of $\mathbb{R}^4$ defined by

$$\begin{aligned} U &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2,\ x_3 = x_4\} \\ V &= \text{span}(\{(1,0,0,1), (0,1,1,0)\}) \\ W &= \{(0, x, 0, y) : x, y \in \mathbb{R}\}. \end{aligned}$$

(a) Is $U + V$ a direct sum of U and V?

Solution: No. Notice that

$$(1,1,1,1) \in U$$

and

$$(1,1,1,1) \in V \qquad (\text{since } (1,1,1,1) = (1,0,0,1) + (0,1,1,0)).$$

Therefore $(1,1,1,1) \in U \cap V$, so $U \cap V \neq \{(0,0,0,0)\}$.

(b) Is $V + W$ a direct sum of V and W?

Solution: Yes. Let $(0, x, 0, y) \in W$. If $(0, x, 0, y)$ is also in U, then

$$0 = x_1 = x_2 = x \qquad \text{and} \qquad 0 = x_3 = x_4 = y.$$

Therefore $V \cap W = \{(0,0,0,0)\}$.

2. Let

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2,\ x_4 = x_1 + x_3\}.$$

Find a subspace W of $\mathbb{R}^4$ such that $V \oplus W = \mathbb{R}^4$.

Solution: First find a basis for V:

$$\begin{aligned} V &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2,\ x_4 = x_1 + x_3\} \\ &= \{(x_1, x_1, x_3, x_4) \in \mathbb{R}^4 : x_4 = x_1 + x_3\} \\ &= \{(x_1, x_1, x_3, x_1 + x_3) \in \mathbb{R}^4\} \\ &= \{x_1(1,1,0,1) + x_3(0,0,1,1) : x_2, x_3 \in \mathbb{R}\} \end{aligned}$$

so $\{(1,1,0,1), (0,0,1,1)\}$ is a basis for V. Now form the matrix A using the basis vectors as rows:

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Notice that A is already in reduced row echelon form. Find the non-pivot columns of RREF(A):

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
$$\quad\ \ \uparrow \quad\ \ \uparrow$$

These are columns 2 and 4. Let W be the span of standard basis vectors with a 1 in the coordinates corresponding to the non-pivot columns:

$$W = \text{span}(\{e_2, e_4\}) = \text{span}(\{(0,1,0,0), (0,0,0,1)\}) = \{(0, x, 0, y) : x, y \in \mathbb{R}\}$$

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We now need to check that $V \cap W = \{(0,0,0,0)\}$. Let $(x_1, x_2, x_3, x_4) \in V \cap W$. Then

$$\begin{aligned} x_1 = x_3 = 0 &\qquad \text{since } (x_1, x_2, x_3, x_4) \in W, \\ x_2 = x_1 = 0 &\qquad \text{since } (x_1, x_2, x_3, x_4) \in V, \\ x_4 = x_1 + x_3 = 0 &\qquad \text{since } (x_1, x_2, x_3, x_4) \in V. \end{aligned}$$

Therefore $V \cap W = \{(0,0,0,0)\}$, so $U + W = U \oplus W$ is a direct sum of U and W. To see that $U \oplus W = \mathbb{R}^4$, notice that

$$\dim(U \oplus W) = \dim(U) + \dim(W) = 2 + 2 = 4 = \dim(\mathbb{R}^4)$$

Since $U \oplus W$ is a subspace of $\mathbb{R}^4$ of equal dimension, $U \oplus W = \mathbb{R}^4$.

3. Find a vector space V with subspaces U_1, U_2, and W such that

$$U_1 \oplus W = U_2 \oplus W$$

but $U_1 \neq U_2$.

Solution: Let

$$\begin{aligned} V &= \mathbb{R}^2 \\ U_1 &= \{(x, 0) : x \in \mathbb{R}\} \\ U_2 &= \{(0, y) : y \in \mathbb{R}\} \\ W &= \{(z, z) : z \in \mathbb{R}\}. \end{aligned}$$

It is not hard to check that

$$U_1 + W = \operatorname{span}\{(1,0), (1,1)\} = \mathbb{R}^2$$

and

$$U_2 + W = \operatorname{span}\{(0,1), (1,1)\} = \mathbb{R}^2$$

so $U_1 + W = U_2 + W$.

Now notice that if $(a, b) \in U_1 \cap W$ then $b = 0$ (since $(a, b) \in U_1$) and $a = b = 0$ (since $(a, b) \in W$). Therefore $U_1 + W$ is a direct sum of $U_1 + W$. A similar argument shows that $U_1 + W$ is a direct sum of U_1 and W.

Finally, notice that $U_1 \neq U_2$, since for example $(1, 0) \in U_1$ but $(1, 0) \notin U_2$.

4. Consider $\mathbb{C}$ as a vector space over the field $F = \mathbb{C}$. Prove that the map $f \colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = \overline{z}$ is not linear.

Solution: It can be checked that f is additive. Therefore we want to find $z \in \mathbb{C}$ and $\lambda \in F = \mathbb{C}$ such that $f(\lambda z) \neq \lambda f(z)$. Let

$$z = 1 \qquad \text{and} \qquad \lambda = i$$

Then

$$f(\lambda z) = f(i) = -i$$

and

$$\lambda f(z) = i \cdot f(1) = i \cdot 1 = i,$$

so $f(\lambda z) \neq \lambda f(z)$.