MTH 342 Worksheet 7

Week 6 – 11/06/2019

Name: **Answer Key** **Recitation time:** ________

1. Let V be a vector space with subsapces $U + W$. Prove that if $U + W = W$ then $U \subseteq W$.

Solution: We already know that $U \subseteq V + W$. Thus if $U + W = W$, then

$$U \subseteq U + W = W,$$

so $U \subseteq W$.

2. Let

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 = x_3\}$$
$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_3 = x_4,\ 2x_2 = x_4\}.$$

(a) Write $U \cap V$ in set-builder notation.

Solution: To form the intersection, write the $U \cap V$ as a subset of $\mathbb{R}$ with the conditions of both U and V:

$$U \cap V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 = x_3,\ x_1 + x_3 = x_4,\ 2x_2 = x_4\}$$

Since $x_1 = x_2 = x_3$ we can rewrite this as

$$U \cap V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 = x_3,\ 2x_1 = x_4\}$$

(b) Find a basis for $U \cap V$.

Solution: From part (a) we have

$$\begin{aligned} U \cap V &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 = x_3,\ 2x_1 = x_4\} \\ &= \{(x_1, x_1, x_1, 2x_1) : x_1 \in \mathbb{R}\} \\ &= \{x_1(1, 1, 1, 2) : x_1 \in \mathbb{R}\}. \end{aligned}$$

A basis for $U \cap V$ is $\{(1, 1, 1, 2)\}$.

2. (Continued) Let

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 = x_3\}$$
$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_3 = x_4,\ 2x_2 = x_4\}.$$

(c) Find a basis for $U + V$.

Solution: Begin by finding bases for U and V.

$$\begin{aligned} U &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 = x_3\} \\ &= \{(x_1, x_1, x_1, x_4) : x_1, x_4 \in \mathbb{R}\} \\ &= \{x_1(1, 1, 1, 0) + x_4(0, 0, 0, 1) : x_1, x_4 \in \mathbb{R}\} \end{aligned}$$

so a basis for U is $\mathcal{B}_1 = \{(1, 1, 1, 0), (0, 0, 0, 1)\}$.

$$\begin{aligned} V &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_3 = x_4,\ 2x_2 = x_4\} \\ &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_3 = 2x_2 - x_1,\ 2x_2 = x_4\} \\ &= \{(x_1, x_2, 2x_2 - x_1, 2x_2) : x_1, x_2 \in \mathbb{R}\} \\ &= \{x_1(1, 0, -1, 0) + x_2(0, 1, 2, 2) : x_1, x_2 \in \mathbb{R}\} \end{aligned}$$

so a basis for V is $\mathcal{B}_2 = \{(1, 0, -1, 0), (0, 1, 2, 2)\}$.

Now consider the set $\mathcal{B}_1 \cup \mathcal{B}_2$. Form a matrix A with the elements of $\mathcal{B}_1 \cup \mathcal{B}_2$ as columns:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Now find the reduced row echelon form of A:

$$A \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Recall that a pivot element in a matrix is the first nonzero element of a *row*. A pivot *column* is a column containing a pivot element. In this case the pivot columns of RREF(A) are the first three columns. Therefore the first three columns of A form a basis for $U + V$:

$$\mathcal{B} = \{(1, 1, 1, 0), (0, 0, 0, 1), (1, 0, -1, 0)\}.$$

3. Find a vector space V with subspaces U_1, U_2, and W such that

$$U_1 + W = U_2 + W$$

but $U_1 \neq U_2$.

Solution: Let $V = \mathbb{R}$, $W = \mathbb{R}$, $U_1 = \mathbb{R}$, and $U_2 = \{0\}$. Then

$$U_1 + W = \mathbb{R} = U_2 + W$$

but

$$U_1 = \mathbb{R} \neq \{0\} = U_2.$$