MTH 342 Worksheet 4

Week 3 – 10/17/2019

Name: **Answer Key** **Recitation time:** ________

1. Let

$$V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = x_1 - x_2\}.$$

a.) Find a basis B_1 for V. What is the dimension of V?

Solution: We can rewrite V as

$$\begin{aligned} V &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = x_1 - x_2\} \\ &= \{(x_1, x_2, x_1 - x_2) : x_1, x_2, x_3 \in \mathbb{R}\} \\ &= \{(x_1, 0, x_1) + (0, x_2, -x_2) : x_1, x_2, x_3 \in \mathbb{R}\} \\ &= \{x_1(1, 0, 1) + x_2(0, 1, -1) : x_1, x_2, x_3 \in \mathbb{R}\} \end{aligned}$$

So every element of V can be written as a linear combination of $\mathbf{v}_1 = (1, 0, 1)$ and $\mathbf{v}_2 = (0, 1, -1)$. Notice that $\mathbf{v}_1$ and $\mathbf{v}_2$ each satisfy the condition $x_3 = x_1 - x_2$, so $\mathbf{v}_1, \mathbf{v}_2 \in V$. These two facts together show that

$$V = \text{Span}(\{\mathbf{v}_1, \mathbf{v}_2\}).$$

To show $B_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V, we must show that they are linearly independent. Let $c_1, c_2 \in \mathbb{R}$ and consider the equation

$$\begin{gathered} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0} \\ \downarrow \\ c_1(1, 0, 1) + c_2(0, 1, -1) = (0, 0, 0) \\ \downarrow \\ (c_1, c_2, c_1 - c_2) = (0, 0, 0) \end{gathered}$$

In this last equation the first coordinate gives $c_1 = 0$ and the second coordinate gives $c_2 = 0$. Therefore B_1 is linearly independent. Since B_1 has exactly two elements, the dimension of V is 2.

b.) Define $f\colon V \to \mathbb{R}^2$ by

$$f(x_1, x_2, x_3) = (x_2,\ x_1 + x_3).$$

Prove that f is a linear map.

Solution: Let $\mathbf{v}, \mathbf{w} \in V$ and $\alpha \in \mathbb{R}$

- We want to show $f(\mathbf{v} + \mathbf{w}) = f(\mathbf{v}) + f(\mathbf{w})$.
  Let $(x_1, x_2, x_3), (y_1, y_2, y_3) \in V$. Then

$$\begin{aligned} f((x_1, x_2, x_3) + (y_1, y_2, y_3)) &= f(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (x_2 + y_2, x_1 + y_1 + x_3 + y_3) \\ &= (x_2, x_1 + x_3) + (y_2, y_1 + y_3) \\ &= f(x_1, x_2, x_3) + f(y_1, y_2, y_3). \end{aligned}$$

- We want to show $f(\alpha\mathbf{v}) = \alpha f(\mathbf{v})$.
  Let $(x_1, x_2, x_3) \in V$ and $\alpha \in \mathbb{R}$ Then

$$\begin{aligned} f(\alpha(x_1, x_2, x_3)) &= f(\alpha x_1, \alpha x_2, \alpha x_3) \\ &= (\alpha x_2, \alpha x_1 + \alpha x_3) \\ &= \alpha(x_2, x_1 + x_3) \\ &= \alpha f(x_1, x_2, x_3). \end{aligned}$$

c.) Let B_2 be the standard basis for $\mathbb{R}^2$. Find the matrix representation $[f]_{B_2,B_1}$ of f as defined in part (b).

Solution: The standard basis for $\mathbb{R}^2$ is

$$B_2 = \{\mathbf{e}_1, \mathbf{e}_2\} = \{(1,0), (0,1)\}.$$

From part (b) we had a basis for V,

$$B_1 = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(1,0,1), (0,1,-1)\}$$

To find the columns of $[f]_{B_2,B_1}$ we must calculate $f(\mathbf{v}_1)$ and $f(\mathbf{v}_2)$ and write the results in terms of $\mathbf{e}_1$ and $\mathbf{e}_2$.

$$\begin{aligned} f(\mathbf{v}_1) &= f(1,0,1) \\ &= (0,2) \\ &= 0(1,0) + 2(0,1) \\ &= 0\mathbf{e}_1 + 2\mathbf{e}_2 \end{aligned}$$

so the first column of $[f]_{B_2,B_1}$ is

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

$$\begin{aligned} f(\mathbf{v}_2) &= f(0,1,-1) \\ &= (1,-1) \\ &= (1,0) - 1(0,1) \\ &= 1\mathbf{e}_1 + (-1)\mathbf{e}_2 \end{aligned}$$

so the second column of $[f]_{B_2,B_1}$ is

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore

$$[f]_{B_2,B_1} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}.$$

Your answer may be different, depending on your

2. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x + 1$. Is h a linear map? Verify your answer.

Solution: h is **not** a linear map. To prove this, we need a counter show that it is not additive or that it is not multiplicative. To see that h is not additive, compare $h(0+0)$ and $h(0)+h(0)$:

$$h(0+0) = h(0) = 1 \neq 2 = h(0) + h(0)$$

so h does **not** satisfy $h(\mathbf{v} + \mathbf{w}) = f(\mathbf{v}) + f(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in V$

3. Let $\mathbb{P}_n$ be the real vector space of polynomials of degree n or less.

a.) Check that the differentiation map $\frac{d}{dx}\colon \mathbb{P}_2 \to \mathbb{P}_1$ is linear.

Solution: You don't need to show any work here. Just recall that

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

and

$$\frac{d}{dx}[\alpha f(x)] = \alpha \frac{d}{dx}[f(x)]$$

for all differentiable functions $f, g\colon \mathbb{R} \to \mathbb{R}$ and all $\alpha \in \mathbb{R}$.

b.) Let $B_1 = \{1, x, x^2\}$ and $B_2 = \{1, x\}$. Then B_1 is a basis for $\mathbb{P}_2$, and B_2 is a basis for $\mathbb{P}_1$. Find the matrix representation $[\frac{d}{dx}]_{B_2,B_1}$ of the differentiation map.

Solution: To find the columns of $[f]_{B_2,B_1}$ we must apply $\frac{d}{dx}$ to each element of B_1 and write the results in terms of the elements of B_2.

$$\begin{aligned} \frac{d}{dx}[1] &= 0 \\ &= 0(1) + 0(x) \end{aligned}$$

so the first column of $[\frac{d}{dx}]_{B_2,B_1}$ is

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{aligned} \frac{d}{dx}[x] &= 1 \\ &= 1(1) + 0(x) \end{aligned}$$

so the second column of $[\frac{d}{dx}]_{B_2,B_1}$ is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\begin{aligned} \frac{d}{dx}[x^2] &= 2x \\ &= 0(1) + 2(x) \end{aligned}$$

so the third column of $[\frac{d}{dx}]_{B_2,B_1}$ is

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

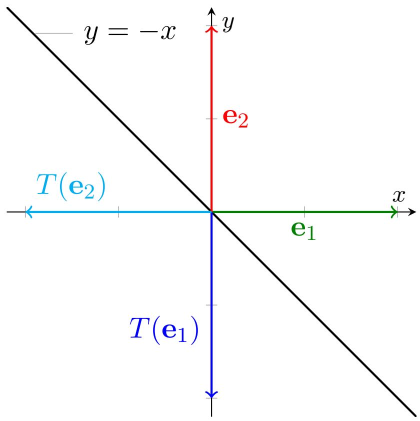
Therefore

$$[\tfrac{d}{dx}]_{B_2,B_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

4. This problem requires you to think geometrically. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects each vector across the line $y = -x$. Find the standard matrix of T.

Solution: By "standard matrix" we mean the matrix $[T]_{B_2,B_1}$ where $B_2 = B_1 = \{\mathbf{e}_1, \mathbf{e}_2\}$ are both the standard basis for $\mathbb{R}^2$.

It may help to visualize the transformations $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$.



$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\mathbf{e}_2 = 0\mathbf{e}_1 + (-1)\mathbf{e}_2,$$
$$T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\mathbf{e}_1 = (-1)\mathbf{e}_1 + 0\mathbf{e}_2$$

The matrix for T is

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$