# MTH 342 Worksheet 3

Week 2 – 10/10/2019

**Name:** Answer Key **Recitation time:** ________

**1.** Determine which of the following sets of functions are linearly independent. If a set is linearly dependent, find a nontrivial linear combination equal to 0.

**a.)** $\{e^x, e^{x+2}\}$

**Solution: Linearly dependent.** Remember that

$$e^{x+2} = e^x e^2 = e^2 \cdot e^x.$$

Therefore

$$\boxed{-e^2(e^x) + (e^{x+2})} = -e^2 \cdot e^x + e^2 \cdot e^x = 0$$

is a linear combination equal to 0 ($c_1 = -e^2$ and $c_2 = 1$).

**b.)** $\{\cos^2(x), \sin^2(x)\}$

**Solution: Linearly independent.** Let $c_1, c_2 \in \mathbb{R}$ be constants and consider the equation

$$c_1 \cos^2(x) + c_2 \sin^2(x) = 0$$

where this is true **for all** $x \in \mathbb{R}$. To prove that the functions are linearly independent, we must show that $c_1 = c_2 = 0$.

If $x = 0$ then the equation becomes

$$c_1(1) + c_2(0) = 0 \qquad \to \qquad c_1 = 0.$$

If $x = \pi/2$ then the equation becomes

$$c_1(0) + c_2(1) = 0 \qquad \to \qquad c_2 = 0.$$

**c.)** $\{\cos^2(x), \sin^2(x), 5\}$

**Solution: Linearly dependent.** Remember the trigonometric identity

$$\cos^2(x) + \sin^2(x) = 1.$$

Subtract 1 from both sides to get

$$\cos^2(x) + \sin^2(x) - 1 = 0.$$

Now we just need to write $-1$ as a $(-\frac{1}{5})5$:

$$\boxed{\cos^2(x) + \sin^2(x) + (-\tfrac{1}{5})5} = 0$$

($c_1 = 1$, $c_2 = 1$, and $c_3 = -\frac{1}{5}$).

**2.** Let $U$ be the subspace of $\mathbb{R}^3$ defined by

$$U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = x_1 + x_2\}.$$

Find a basis for $U$.

**Solution:** The vectors

$$\begin{aligned} \mathbf{v}_1 &= (1, 0, 1) \\ \mathbf{v}_2 &= (0, 1, 1) \end{aligned}$$

form a basis for $U$. To prove this, we need to prove two facts:

(i) $\mathbf{v}_1, \mathbf{v}_2 \in U$ and every element of $U$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$
*proof.* It is easy to see that $\mathbf{v}_1, \mathbf{v}_2 \in U$. Now any element of $U$ can be written as

$$\begin{aligned} (x_1, x_2, x_1 + x_2) &= (x_1, 0, x_1) + (0, x_2, x_2) \\ &= x_1(1, 0, 1) + x_2(0, 1, 1) \end{aligned}$$

for some $x_1, x_2 \in \mathbb{R}$, which is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$

(ii) $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.
*proof.* Consider the equation

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= \mathbf{0} \\ &\downarrow \\ c_1(1, 0, 1) + c_2(0, 1, 1) &= (0, 0, 0) \\ &\downarrow \\ (c_1, 0, c_1) + (0, c_2, c_2) &= (0, 0, 0) \\ &\downarrow \\ (c_1, c_2, c_1 + c_2) &= (0, 0, 0) \end{aligned}$$

The equation in the first coordinate shows that

$$c_1 = 0.$$

Similarly, the equation in the second coordinate shows that

$$c_2 = 0.$$

Therefore $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

**3.** Let $V$ and $W$ be vector spaces over a field $F$ and let $f : V \to W$ be a linear map. Prove that the set

$$\{\mathbf{v} \in V : f(\mathbf{v}) = \mathbf{0}\}$$

is a subspace of $V$ (this is called the **null space of $\boldsymbol{f}$**).

**Solution:** Call this set $N$. To show that $N$ is a subspace of $V$ we need to show that it is closed under addition and scalar multiplication and that it contains $\mathbf{0}$.

(i) Closed under addition: Suppose $\mathbf{v}, \mathbf{w} \in N$. Then

$$\begin{aligned} f(\mathbf{v}+\mathbf{w}) &= f(\mathbf{v}) + f(\mathbf{w}) && \text{(since } f \text{ is linear)} \\ &= \mathbf{0} + \mathbf{0} && \text{(since } \mathbf{v}, \mathbf{w} \in N\text{)} \\ &= \mathbf{0}, \end{aligned}$$

so $\mathbf{v} + \mathbf{w} \in N$.

(ii) Closed under scalar multiplication: Suppose $\mathbf{v} \in N$ and $\alpha \in F$. Then

$$\begin{aligned} f(\alpha\mathbf{v}) &= \alpha f(\mathbf{v}) && \text{(since } f \text{ is linear)} \\ &= \alpha\mathbf{0} && \text{(since } \mathbf{v} \in N\text{)} \\ &= \mathbf{0}, \end{aligned}$$

so $\alpha\mathbf{v} \in N$.

(iii) $N$ contains the zero vector:

$$\begin{aligned} f(\mathbf{0}) &= f(\mathbf{0} + \mathbf{0}) \\ &= f(\mathbf{0}) + f(\mathbf{0}). \end{aligned}$$

Subtracting $f(\mathbf{0})$ from both sides we get

$$\mathbf{0} = f(\mathbf{0})$$

so $\mathbf{0} \in N$.

Therefore $N$ is a subspace of $V$.

**4.** Let $V$ be a vector space and suppose $\mathbf{v}_1, \mathbf{v}_2 \in V$ are linearly independent. Let $\mathbf{w}_1, \mathbf{w}_2 \in V$ such that $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{v}_1$ and $\mathbf{w}_1 - \mathbf{w}_2 = \mathbf{v}_2$. Prove that $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent.

**Solution:** This problem may have been a bit too difficult for this worksheet. Here's one way to prove this:

First, we can solve for $\mathbf{w}_1$ and $\mathbf{w}_2$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$. Consider the given equations

$$\begin{aligned} \mathbf{w}_1 + \mathbf{w}_2 &= \mathbf{v}_1 \\ \mathbf{w}_1 - \mathbf{w}_2 &= \mathbf{v}_2 \end{aligned}$$

Adding these equations gives

$$2\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2 \qquad \rightarrow \qquad \mathbf{w}_1 = \tfrac{1}{2}\mathbf{v}_1 + \tfrac{1}{2}\mathbf{v}_2.$$

Similarly, subtracting the second equation from the first gives

$$2\mathbf{w}_2 = \mathbf{v}_1 - \mathbf{v}_2 \qquad \rightarrow \qquad \mathbf{w}_2 = \tfrac{1}{2}\mathbf{v}_1 - \tfrac{1}{2}\mathbf{v}_2.$$

Now we can show that $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent.

Let $c_1$ and $c_2$ be constants and consider the equation

$$c_1\mathbf{w}_2 + c_2\mathbf{w}_2 = 0$$

We want to show that $c_1 = c_2 = 0$.

$$\begin{aligned} c_1\mathbf{w}_2 + c_2\mathbf{w}_2 &= 0 \\ &\downarrow \\ c_1\left(\tfrac{1}{2}\mathbf{v}_1 + \tfrac{1}{2}\mathbf{v}_2\right) + c_2\left(\tfrac{1}{2}\mathbf{v}_1 - \tfrac{1}{2}\mathbf{v}_2\right) &= 0 \\ &\downarrow \\ \tfrac{1}{2}c_1\mathbf{v}_1 + \tfrac{1}{2}c_1\mathbf{v}_2 + \tfrac{1}{2}c_2\mathbf{v}_1 - \tfrac{1}{2}c_2\mathbf{v}_2 &= 0 \\ &\downarrow \\ \left(\tfrac{1}{2}c_1 + \tfrac{1}{2}c_2\right)\mathbf{v}_1 + \left(\tfrac{1}{2}c_1 - \tfrac{1}{2}c_2\right)\mathbf{v}_2 &= 0 \end{aligned}$$

Now since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent we must have

$$\begin{aligned} \tfrac{1}{2}c_1 + \tfrac{1}{2}c_2 &= 0 \\ \tfrac{1}{2}c_1 - \tfrac{1}{2}c_2 &= 0 \end{aligned}$$

This system of equations can be solved easily. Add the top equation to the bottom equation to get

$$c_1 = 0$$

Then the first equation gives

$$\tfrac{1}{2}c_1 + \tfrac{1}{2}c_2 = 0 \qquad \rightarrow \qquad 0 + \tfrac{1}{2}c_2 = 0 \qquad \rightarrow \qquad c_2 = 0.$$

**5.** Determine (with proof) which of the following maps are linear.

**a.)** $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

**Solution: $f$ is not linear.** Notice that $f(1) = 1$ and $f(2) = 4$, but

$$f(1+2) = f(3) = 9 \neq 5 = f(1) + f(2).$$

**b.)** $A\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $A(x_1, x_2) = x_1 + x_2$.

**Solution: $A$ is linear.** Let $(x_1, x_2), (z_1, z_2) \in \mathbb{R}^2$ and let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} A((x_1, x_1) + (z_1, z_2)) &= A(x_1 + z_1, x_2 + z_2) \\ &= (x_1 + z_1) + (x_2 + z_2) \\ &= (x_1 + x_2) + (z_1 + z_2) \\ &= A(x_1, x_2) + A(z_1, z_2) \end{aligned}$$

and

$$\begin{aligned} A(\alpha(x_1, x_1)) &= A(\alpha x_1, \alpha x_2) \\ &= \alpha x_1 + \alpha x_2 \\ &= \alpha(x_1 + x_2) \\ &= \alpha A(x_1, x_2) \end{aligned}$$

so $A$ is linear.