MTH 342 Worksheet 1

Week 1 – 9/26/2019

Name: **Answer Key**

The solutions given in this answer key are often more wordy than necessary.

1. Check if the following statements are true or false. If true, give a brief explanation. If false, give a counterexample.

   (i) An $m \times n$ matrix has m rows and n columns.
   Solution: True. This is simply by the definition of an $m \times n$ matrix.

   (ii) If f and g are polynomials of degree n, then $f + g$ is also a polynomial of degree n.
   Solution: False.
   Let f and g be the following degree 3 polynomials:

   $$f(x) = 3x^3 + x - 1$$
   $$g(x) = -3x^3 + x^2 + x.$$

   Then

   $$f(x) + g(x) = x^2 + 2x - 1$$

   is a degree 2 polynomial.

   What is true: If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree less than or equal to n.

   (iii) A linear system of 2 equations and 3 unknowns always has infinitely many solutions.
   Solution: False.
   The system

   $$\begin{cases} x + y + z &= 3 \\ x + y + z &= 4 \end{cases}$$

   cannot have any solutions. If it did have a solution, then

   $$3 = x + y + z = 4$$

   but this is clearly false, since $3 \neq 4$.

   What is true: A linear system of 2 equations and 3 unknowns has either no solutions or infinitely many solutions.

   (iv) A linear system of 3 equations and 2 unknowns is always inconsistent (i.e. has no solutions).
   Solution: False.
   The system

   $$\begin{cases} x + y &= 5 \\ 2x - y &= 1 \\ -x + y &= 1 \end{cases}$$

   has the solution $x = 2$, $y = 3$ (you can check this by plugging these values into the system).

   What is true: A linear system of 3 equations and 2 unknowns may have no solutions, exactly one solution, or infinitely many solutions.

2. Evaluate the following matrix operations.

(i) $\begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

Solution: Recall matrix multiplication. Multiplying an $m \times n$ matrix A by an $n \times p$ matrix B gives an $m \times p$ matrix C. The entry in the ith row and jth column of C is obtained by multiplying term-by-term the entries of the ith row of A and the jth column of B, and summing these m products.

$$\begin{aligned} \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} &= \begin{bmatrix} (1)(2) + (-1)(3) + (0)(1) \\ (2)(2) + (-1)(3) + (2)(1) \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 3 \end{bmatrix} \end{aligned}$$

(ii) $\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \end{bmatrix}$

Solution: You cannot add matrices of different dimensions.

(iii) $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \end{bmatrix}$

Solution:

$$\begin{aligned} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \end{bmatrix} &= \begin{bmatrix} (1)(2) & (1)(0) \\ (-1)(2) & (1)(0) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix} \end{aligned}$$

(iv) $(1+i) \begin{bmatrix} 1-2i & 2+i \\ 0 & 1-i \end{bmatrix}$

Solution: Recall that multiplying a matrix by a scalar (i.e. a number) simply multiplies each entry in the matrix by that scalar.

$$\begin{aligned} (1+i) \begin{bmatrix} 1-2i & 2+i \\ 0 & 1-i \end{bmatrix} &= \begin{bmatrix} (1+i)(1-2i) & (1+i)(2+i) \\ (1+i)(0) & (1+i)(1-i) \end{bmatrix} \\ &= \begin{bmatrix} 1-2i+i-2i^2 & 2+i+2i+i^2 \\ 0 & 1-i+i-i^2 \end{bmatrix} \\ &= \begin{bmatrix} 1-i-2(-1) & 2+3i+(-1) \\ 0 & 1-(-1) \end{bmatrix} \\ &= \begin{bmatrix} 3-i & 1+3i \\ 0 & 2 \end{bmatrix} \end{aligned}$$

3. Give an example of a fourth degree polynomial with four distinct complex roots, two of which are real, the other two of which are not.

Solution: This problem can be solved by writing the polynomial in factored form:

$$(x - r_1)(x - r_2)(x - r_3)(x - r_4)$$

where $r_1, \ldots, r_4$ are the roots of the polynomial. For example, the polynomial

$$(x - 1)(x + 2)(x + 1 - i)(x - 2 - 3i)$$

has real roots 1 and -2 and non-real roots $-1 + i$ and $2 + 3i$.

Another particularly simple solution (this time written in standard form) is the polynomial $x^4 - 1$ which has roots 1, -1, i, and $-i$. You can check this by plugging each root into the polynomial and verifying that you get 0.

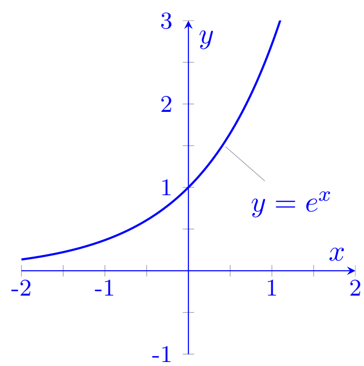
4. Consider the set $S = \{(x, e^x) : x \in \mathbb{R}\}$.

   (i) Visualize S.

   (ii) Is S closed under addition or scaling?

   (iii) Verify your answer algebraically and geometrically.

Solution: The set S can be visualized as the graph of the function $y = e^x$ in the x-y plane:



S is not closed under addition or scaling. To prove this algebraically, consider the element $(0, e^0) = (0, 1) \in S$. Then

$$(0, 1) + (0, 1) = 2 \cdot (0, 1) = (0, 2) \notin S.$$

S is not closed under addition, because $(0, 1) + (0, 1) \notin S$. S is also not closed under scaling, because $2 \cdot (0, 1) \notin S$. This can be verified visually by seeing that the point $(0, 2)$ does not lie on the curve $y = e^x$.

Geometrically, addition can be viewed as a translation in the plane (i.e., a shift). For example, adding $(0, 1)$ (which is an element of S) shifts a point up by 1. Notice that shifting any point on the curve $y = e^x$ up by 1 gives a new point that is not on the curve. Since $(0, 1)$ is an element of S, this means that S is not closed under addition.

In the x-y plane, scalar multiplication can be viewed geometrically as stretching from or compressing towards the origin. Multiplication by a negative number causes a flip across the origin. Notice that scalar multiplication by 0 always gives the point $(0, 0)$ (i.e, it compresses all points down to the origin). In particular, $0 \cdot (x, e^x) = (0, 0)$. Since $(0, 0)$ is not on the curve $y = e^x$, this shows that S is not closed under scalar multiplication.

5. Solve the following linear system using the row reduction method (Gauss or Gauss-Jordan):

$$\begin{cases} 3x + 2y - z &= 1 \\ x - y + 2z &= -1 \end{cases}$$

Solution: There are many ways to do this. Here is one:

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 1 & -1 & 2 & -1 \end{array}\right] \xrightarrow{\text{swap } R_1 \text{ and } R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 3 & 2 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 - 3R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 5 & -7 & 4 \end{array}\right]$$

$$\xrightarrow{5R_1 \to R_1} \left[\begin{array}{ccc|c} 5 & -5 & 10 & -5 \\ 0 & 5 & -7 & 4 \end{array}\right]$$

$$\xrightarrow{R_1 + R_2 \to R_1} \left[\begin{array}{ccc|c} 5 & 0 & 3 & -1 \\ 0 & 5 & -7 & 4 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & -\frac{1}{5} \\ 0 & 5 & -7 & 4 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & -\frac{7}{5} & \frac{4}{5} \end{array}\right]$$

$$\to \begin{cases} x + \frac{3}{5}z = -\frac{1}{5} \\ y - \frac{7}{5}z = \frac{4}{5} \end{cases}$$

so the solutions are given by

$$x = -\frac{1}{5} - \frac{3}{5}z,$$
$$y = \frac{4}{5} + \frac{7}{5}z,$$

with z an arbitrary scalar.