Lecture 9 (10/14/2019)

$f: V \to W$ linear.

If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, we know that f can be represented by a matrix

$$A = \begin{bmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{bmatrix}$$

Moreover, $f(x) = Ax$.

How about the general case when V and W are not necessarily $\mathbb{R}^n$? We indeed have a very similar result.

First, one needs to specify a basis for V an W.

V has basis $B_1 = \{v_1, v_2, \dots, v_n\}$.

W has basis $B_2 = \{w_1, w_2, \dots, w_m\}$.

Strictly speaking, B_1 and B_2 are lists (with fixed order). When one changes the order of vectors in a basis, the coordinates of a vector are also changed.

$$[f]_{B_2,B_1} \stackrel{\text{def}}{=} \begin{bmatrix} | & | & & | \\ [f(v_1)]_{B_2} & [f(v_2)]_{B_2} & \cdots & [f(v_n)]_{B_2} \\ | & | & & | \end{bmatrix}$$

Then we have:

$$[f(v)]_{B_2} = [f]_{B_2,B_1} [v]_{B_1}. \qquad (*)$$

Recall: a colum vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is called coordinate vector of $v \in V$ with respect to basis B_1 if

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n.$$

One denotes
$$[v]_{B_1} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Ex:

$V = \mathbb{R}^3$ has basis $B = \{v_1, v_2, v_3\}$ where

$v_1 = (1, 2, 3)$

$v_2 = (1, 0, 1)$

$v_3 = (0, 2, 1)$

Find coordinate vector of $v = (4, 0, 5)$ in basis B.

We want to solve for c_1, c_2, c_3 such that $v = c_1 v_1 + c_2 v_2 + c_3 v_3$.

We can write this equation in matrix form as follows:

$$\underbrace{\begin{bmatrix} | \\ v \\ | \end{bmatrix}}_{[v]_{B_0}} = \underbrace{\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{[v]_B}$$

B_0 is the standard basis $B_0 = \{e_1, e_2, e_3\}$. Thus,
$$[v]_B = P^{-1}[v]_{B_0}$$

The problem of changing basis in $\mathbb{R}^n$ is in fact a particular case of finding matrix representation of a linear map. Consider the following example.

Suppose B_1 and B_2 are two bases of $\mathbb{R}^n$. Knowing $[v]_{B_1}$. How do we find $[v]_{B_2}$?

Consider the identity map $id : \underbrace{(\mathbb{R}^n, B_1)}_{\substack{\mathbb{R}^n \text{ with} \\ \text{basis } B_1}} \longrightarrow \underbrace{(\mathbb{R}^n, B_2)}_{\substack{\mathbb{R}^n \text{ with} \\ \text{basis } B_2}}$

Apply equation $(*)$: $[v]_{B_2} = [id]_{B_2,B_1}[v]_{B_1}$.

The problem of find $[v]_{B_2}$ becomes the problem of finding the matrix $[id]_{B_2,B_1}$.

<u>Ex</u>: Let $f: \mathbb{R}^3 \to M_{2\times 2}(\mathbb{R})$

$$f(a,b,c) = \begin{bmatrix} a & b \\ a+c & 0 \end{bmatrix}$$

Find a matrix representation of f.

One can check that f is indeed a linear map (by checking additivity and scalar multiplicative). Let us assume that this step is already done.

$$V = \mathbb{R}^3, \quad W = M_{2\times 2}(\mathbb{R}).$$

We need to specify a basis of V and W.

$$B_1 = \{\underbrace{(1,0,0)}_{e_1}, \underbrace{(0,1,0)}_{e_2}, \underbrace{(0,0,1)}_{e_3}\} \quad \text{(standard basis)}$$

The standard basis of $M_{m\times n}(\mathbb{R})$ is $\{E_{ij} : 1 \le i \le m, 1 \le j \le n\}$.

$$E_{ij} = \begin{array}{cl} \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} & \leftarrow i \\ \uparrow & \\ j & \end{array}$$

$M_{m\times n}(\mathbb{R})$ is an mn-dimensional vector space over $\mathbb{R}$.

$W = M_{2\times n}(\mathbb{R})$ is therefore a 4-dimensional vector space with basis

$$B_2 = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_4} \right\}$$

Then

$$[f]_{B_2,B_1} = \begin{bmatrix} | & | & | \\ [f(e_1)]_{B_2} & [f(e_2)]_{B_2} & [f(e_3)]_{B_2} \\ | & | & | \end{bmatrix}$$

(4×3 matrix).

To find the first column, we write $f(e_1)$ as a linear combination of E_1, E_2, E_3, E_4. By the definition of f,

$$f(e_1) = f(1,0,0) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = E_1 + E_3.$$

Thus,

$$[f(e_1)]_{B_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Similarly,

$$f(e_2) = f(0,1,0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_2$$

$$f(e_3) = f(0,0,1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = E_3$$

$$[f(e_2)]_{B_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [f(e_3)]_{B_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore,

$$[f]_{B_2,B_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

See worksheet for another example.