# Lecture 8 (10/11/2019)

$V$ and $W$ are vector spaces over $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. A map $f: V \to W$ is said to be linear if it satisfies two following conditions

- Additive:

$$f(v_1 + v_2) = f(v_1) + f(v_2) \quad \forall v_1, v_2 \in V$$

- Scalar multiplicative:

$$cf(v) = f(cv) \quad \forall c \in \mathbb{R}, v \in V$$

Ex:

1) $F: \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$, $F(u) = xu$
is a linear map (see worksheet).

2) $\det: M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ is not a linear map (see worksheet).

Linear maps provide a means to study vectors spaces. For example, we know that $V_n$, the set of all polynomials of degree $\leq n$, is an $(n+1)$-dimensional vector space. In some sense, it looks like $\mathbb{R}^{n+1}$. Indeed, consider the map

$$F: \mathbb{R}^{n+1} \longrightarrow V_n$$

$$F(a_0, a_1, \ldots, a_n) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

One can check that $F$ is a linear map. Thanks to $f$, adding two vectors in $\mathbb{R}^{n+1}$ reflects adding two polynomials. Similarly for scaling.

Recall that a linear map $f: \mathbb{R}^n \to \mathbb{R}^m$ is represented by an $m \times n$ matrix.

$$A = \underbrace{\begin{bmatrix} | & | & & | \\ f(e_1) & f(e_2) & \ldots & f(e_n) \\ | & | & & | \end{bmatrix}}_{n \text{ columns}} \Bigg\} \; m \text{ rows because each vector is of length } m.$$

One can ask how to represent a linear map $f: V \to W$.
We mimick the idea in $\mathbb{R}^n$. Let us fix a basis of $V$ and a basis of $W$.

$V$ has a basis $B_1 = \{v_1, v_2, \dots, v_n\}$.

$W$ has a basis $B_2 = \{w_1, w_2, \dots, w_m\}$.

Then $f$ is represented by matrix

$$[f]_{B_2, B_1} = \begin{bmatrix} | & | & & | \\ [f(v_1)]_{B_2} & [f(v_2)]_{B_2} & \cdots & [f(v_n)]_{B_2} \\ | & | & & | \end{bmatrix}$$

where $[f(v_i)]_{B_2}$ is the coordinate of vector $f(v_i)$ in basis $B_2$.
We will explain more on the notations and give some examples next time.