Lecture 7 (10/9/2019)

Ex: Let V be the set of all polynomials with coefficients in $\mathbb{R}$.
Then V is an infinite dimensional vector space over $\mathbb{R}$.
Why? Suppose by contradiction that V is finite dimensional.
Then .... (excercise).

Ex: Let V be the vector space of all symmetric 2×2 matrices.
(Why is it a vector space?)
Find a basis of V.
A vector in V is of the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{with } b = c$$

which can be rewritten in a condition-free manner as

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \quad \text{where } a, b, d \in \mathbb{R}$$

This vector can be expressed as a sum

$$A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

then a combination of sum and scaling

$$A = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

In other words, A is a linear combination of

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Because any vector in V is a linear combination of E_1, E_2, E_3,
these vectors span V.

To check whether they form a basis of V, we need to check if they are linearly independent. Let us solve for $c_1, c_2, c_3 \in \mathbb{R}$ from the equation

$$c_1 E_1 + c_2 E_2 + c_3 E_3 = 0$$

the zero matrix.

This is equivalent to

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We see that
$$LHS = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix}$$

$LHS = RHS$ if and only if $c_1 = c_2 = c_3 = 0$.

Therefore, E_1, E_2, E_3 are linearly ind.

$\{E_1, E_2, E_3\}$ is a basis of V and $\dim_{\mathbb{R}} V = 3$.

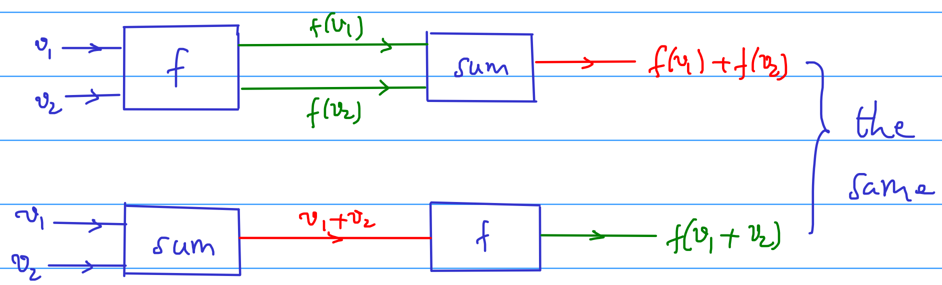
Linear maps

Let V and W be vector spaces over a field $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. A map

$$f : V \to W$$

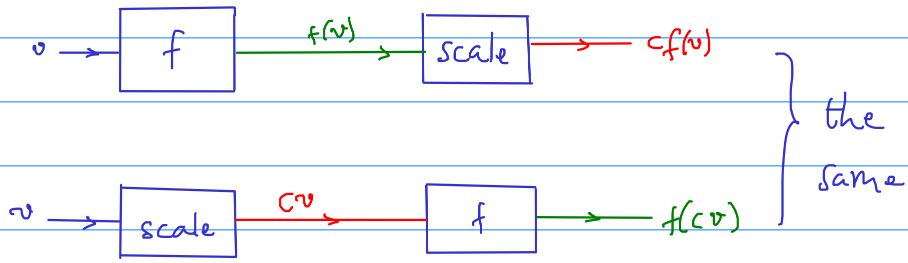
is said to be linear if it satisfies two conditions:

- Additive: $f(v_1 + v_2) = f(v_1) + f(v_2) \quad \forall v_1, v_2 \in V$



(the order of applying f and summing doesn't matter).

- Scalar multiplicative: $f(cv) = cf(v) \quad \forall v \in V, c \in F$



(the order of applying f and scaling doesn't matter).

Ex: the constant function 0 is a linear map.

Ex:

$V = \{$ differentiable functions from $\mathbb{R}$ to $\mathbb{R}\}$

$W = \mathbb{R}^{\mathbb{R}}$

The map $F: V \to W$, $F(u) = u'$ is linear.

Ex:

$V = \{$ continuous functions from $[0,1]$ to $\mathbb{R}\}$

$W = \mathbb{R}$

The map $I: V \to W$, $\quad I(f) = \int_0^1 f(x)\,dx$

is linear.

More examples are on the worksheet.