Lecture 6 (10/7/2019)

Recall the definition of linear independence:

V.... vector space over $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$

definition: $v_1, v_2, \dots, v_n \in V$ are said to be lin. ind. if the equation

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

zero vector in V

has only one solution: $c_1 = c_2 = \dots = c_n = 0$.

Like in $\mathbb{R}^n$, we have an equivalent definition of linear independence as follows:

equivalent definition: $v_1, v_2, \dots, v_n$ are lin. ind. if none of them can be written as a linear combination of the others.

Like in $\mathbb{R}^n$, we can define basis of a vector space as follows.

definition: V.... vector space over $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. A subset $S \subset V$ is said to be a basis of V if S is linearly independent and spans V.

An infinite set S is linearly independent if any finite subset of S is linearly independent.

We have an equivalent definition of basis as follows:

equivalent definition: A subset $S \subset V$ is a basis of V if every vector in V is has a unique representation as linear combination of vectors in S.

$\dim_F V = \# S$ (which can be ∞).

Ex: V_n = set of all polynomials with coefficients in F of degree $\leq n$.

This is a vector space over F.

Note: as standard convention, 0 is a polynomial of degree $-\infty$.

Why? Note that an element of V_n is a function from F to F. Thus, $V_n \subset \underbrace{F^F}_{\text{vector space over } F}$

V_n satisfies 3 properties:

- $0 \in V_n$
- closed under addition: because the sum of two poly. of degree $\le n$ is still a poly. of degree $\le n$.
- closed under scaling: because the product of a poly. by a constant is still a poly. of degree $\le n$.

What is a basis of V_n?

A vector in V_n is of the form

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad \text{where } a_0, a_1, \dots, a_n \in F.$$

If we denote

$$y_n = x^n, \quad y_{n-1} = x^{n-1}, \quad \dots, \quad y_1 = x, \quad y_0 = 1$$

Then $y_n, y_{n-1}, \dots, y_1, y_0$ belong to V_n and

$$y = a_n y_n + a_{n-1} y_{n-1} + \cdots + a_1 y_1 + a_0 y_0$$

In other words, y is a linear combination of $y_0, y_1, \dots, y_n$.

In other words, $y_0, y_1, \dots, y_n$ span V_n.

Are $y_0, y_1, y_2, \dots, y_n$ linearly independent?

We will solve the equation

$$c_0 y_0 + c_1 y_1 + \cdots + c_n y_n = 0 \qquad (*)$$

for $c_0, c_1, \dots, c_n$. If this equation turns out to have only one solution $c_0 = c_1 = \cdots = c_n = 0$ then we conclude that $y_0, \dots, y_n$ are linearly independent.

Equation (*) can be rewritten as

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0 \qquad \forall x \in F$$

We want to show that $c_0, c_1, \ldots, c_n = 0$. Suppose by contradiction that not all $c_0, c_1, \ldots, c_n$ are equal to zero. Then LHS is a nonzero polynomial. Then it has at most n roots. This is a contradiction because every $x \in F$ is a root of LHS. Therefore, $c_0, c_1, \ldots, c_n = 0$.