Lecture 5 (10/4/2019)

Last time, we learn an alternative way to check if a set is a vector space over a field of numbers $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ (rather than checking all vector space axioms).

To show that a set W is a vector space over F, we look for a vector space V over F that contains W. Then check 3 properties:

- $0 \in W$
- W is closed under addition
- W is closed under scaling.

Ex: Let W be the set of all twice-differentiable functions $y: \mathbb{R} \to \mathbb{R}$ such that $y'' + y = 0$. The addition and scaling of function are understood in usual sense (i.e. addition pointwise and scaling pointwise). Show that W is a vector space over $\mathbb{R}$.

We see that W is a subset of $V = \mathbb{R}^{\mathbb{R}}$ (the set of all functions from $\mathbb{R}$ to $\mathbb{R}$). We know that V is a vector space over $\mathbb{R}$. (Recall that F^S is always a vector space over F.)

Thus we only need to check 3 properties:

- Check if $\underset{\substack{\uparrow \\ \text{the constant funtion } 0}}{0} \in W$:

The constant function $f \equiv 0$ belongs to W because it is twice differentiable and satisfies $f'' + f = 0$.

- Check if W is closed under addition.

Let $f, g \in W$. Want to show $\underbrace{f+g}_{h} \in W$.

That is to show $\begin{cases} h \text{ is twice differentiable} \\ h'' + h = 0 \end{cases}$

Because f and g are twice differentiable, so is h.
(f'' and g'' exists, so $(f+g)''$ also exists.)
How to show $h''+h=0$?
Fix $x \in \mathbb{R}$. Want to show $h''(x)+h(x)=0$.
We have

$$h''(x)+h(x) = (f''(x)+g''(x)) + (f(x)+g(x))$$
$$= (f''(x)+f(x)) + (g''(x)+g(x))$$

Because $f, g \in W$,

$$f''(x)+f(x) = g''(x)+g(x) = 0$$

Therefore,

$$h''(x)+h(x) = 0.$$

- Check if W is closed under scaling.
  Let $f \in W$ and $c \in \mathbb{R}$. Want to show $\underbrace{cf}_{h} \in W$.

  That is to show $\begin{cases} h \text{ is twice differentiable} \\ h''+h=0 \end{cases}$

  The rest of the argument is similar to the above. ∎

One can notice that $y_1 = \sin x$ and $y_2 = \cos x$ are two solutions of the differential eq. $y''+y=0$. They are elements (vectors) in the vector space W. We can talk about their linear independence (or dependence).

Def:

Let V be a vector space over $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Vectors $v_1, v_2, \dots, v_n \in V$ are said to be <u>linearly independent</u> if the equation $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ implies $c_1 = c_2 = \cdots = c_n = 0$.
If $v_1, v_2, \dots, v_n$ are not lin. ind., they are said to be lin. dependent.

A vector of the form $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ is said to be a linear combination of $v_1, v_2, \ldots, v_n$.

Ex:

The vectors $y_1 = \sin x$, $y_2 = \cos x$ as above are linearly independent.

Why? Consider the equation $c_1 y_1 + c_2 y_2 = 0$

This equation means

$$c_1 \sin x + c_2 \cos x = 0 \quad \forall x \in \mathbb{R}$$

Take $x = 0$: $\quad c_1(0) + c_2(1) = 0$.

Then $c_2 = 0$.

Take $x = \pi/2$: $\quad c_1(1) + c_2(0) = 0$

Then $c_1 = 0$.

Because $c_1 = c_2 = 0$, y_1 and y_2 are linearly independent.

*Observation: It is very hard for a list of functions to be linearly dependent.

More examples are on the worksheet.