Lecture 4 (10/2/2019)

* An element of a vector space V is called a vector.

Two useful vector spaces are the following:

1) The set of $m \times n$ matrices with coefficients in a field $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, denoted by $M_{m\times n}(F)$ is a vector space over F. The zero element zero matrix $\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}$

Ex: $M_{2\times 3}(\mathbb{Q})$ is a vector space (over $\mathbb{Q}$) of 2×3 matrices with coefficients in $\mathbb{Q}$.

$M_{n\times 1}(\mathbb{R})$ and $M_{1\times n}(\mathbb{R})$, both of which can be "identified" as $\mathbb{R}^n$, are vector spaces over $\mathbb{R}$.

2) Let S be a nonempty set. The set of all functions from S to a field $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, denoted by F^S, is a vector field over F. The addition and scalar multiplication are defined as follows:

- For $f, g : S \to F$, function $f+g$ is defined by
$$(f+g)(x) = f(x) + g(x) \quad \forall x \in S$$
- For any $c \in F$ and $f: S \to F$, function cf is defined by
$$(cf)(x) = cf(x) \quad \forall x \in S.$$

We see that the zero element of F^S is the constant function $f \equiv 0$.

Ex: the set of all functions from $[0,1]$ to $\mathbb{R}$ is a vector space over $\mathbb{R}$, denoted by $\mathbb{R}^{[0,1]}$.

Ex: the set of all functions from $\mathbb{N}$ to $\mathbb{C}$ is a vector space over $\mathbb{C}$, denoted by $\mathbb{C}^{\mathbb{N}}$. Note that such a function can be viewed as a sequence $a_n = f(n)$. Thus, the set of all sequences (a_n) with complex coefficients is a vector space over the field of complex numbers.

* Last time, we verify the set

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{C},\ a + 2b = 0 \right\}$$

to be a vector space over $\mathbb{C}$ by checking all axioms of vector space (A0), ..., (D2). Now we introduce another way to verify that V is a vector space over $\mathbb{C}$. This involves the notion of <u>subspace</u>.

Def:

Let V be a vector space over F. A set $W \subset V$ is called a subspace of V if W itself is a vector space over F.

Ex:

- $[0,1]$ is a subset of $\mathbb{R}$, but not a subspace of $\mathbb{R}$ because $\underbrace{3}_{\mathbb{R}} \cdot \underbrace{\tfrac{1}{2}}_{[0,1]} = \tfrac{3}{2} \notin [0,1]$.

- $\mathbb{Q}$ is a subset of $\mathbb{R}$, but not a subspace of $\mathbb{R}$ because $\underbrace{\sqrt{3}}_{\mathbb{R}} \cdot \underbrace{1}_{\mathbb{Q}} = \sqrt{3} \notin \mathbb{Q}$.

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ is a subset of $\mathbb{Q}$, but not a subspace of $\mathbb{Q}$ because $\underbrace{\tfrac{7}{3}}_{\mathbb{Q}} \times \underbrace{2}_{\mathbb{N}} \notin \mathbb{N}$.

- The set $\{(a, 0) : a \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$ over $\mathbb{R}$.

How to check if a subset is a subspace?

Theorem: Let V be a vector space over F, and W be a subset of V. Then W is a subspace of V if and only if the 3 following properties are satisfied:

(1) W contains the zero vector of V:

$$0 \in W$$

(2) W is closed under addition:

If $u, v \in W$ then $u + v \in W$.

(3) W is closed under scaling:

If $c \in F$ and $u \in W$ then $cu \in W$.

* Comment:

The theorem says that, if we know that W is a subset of a vector space V (over F) then: to show W is a vector space over F, we only need to check (A0) and (S0).

Let's revisit the example

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{C},\ a + 2b = 0 \right\}$$

We see that V is a subset of $M_{2\times2}(\mathbb{C})$. To show that V is a vector space over $\mathbb{C}$ (or equivalently, a subspace of $M_{2\times2}(\mathbb{C})$), it suffices to check the following:

- Check (1):

  The zero element of $M_{2\times2}(\mathbb{C})$, which is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

  belongs to V because $0 + 2(0) = 0$.

- Check (2):

  Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

be two elements of V.

We want to check if $A+B \in V$.

$$A+B = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

We want to check if $a+e+2(b+f)=0$ $\quad (*)$

Because $A \in V$, $a+2b=0$.

Because $B \in V$, $e+2f=0$.

Summing these equations, we get $\underbrace{a+2b+e+2f}_{a+e+2(b+f)}=0$

Thus, $(*)$ is true.

- Check (3):

  Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an element of V

  and $\alpha \in \mathbb{C}$. We want to check if $\alpha A \in V$.

$$\alpha A = \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$$

  We want to check if $\alpha a + 2\alpha b = 0$.

  In other words, we want to check if $\alpha(a+2b)=0$. $\quad (**)$

  Because $A \in V$, $a+2b=0$.

  Thus, $(**)$ is true.