Lecture 3 (9/30/2019)

Recall: a structure of $(V, +, \cdot, F)$ is a vector space if

(A) addition

- (A0) V is closed under addition
- (A1) commutative
- (A2) associative
- (A3) there is a zero element
- (A4) each element has an inverse

(S) scaling (scalar multiplication)

- (S0) V is closed under scaling
- (S1) associative
- (S2) scaling by 1 doesn't do anything

(I) interaction between addition and scaling

- (I1) distribution
- (I2) distribution

* For short, one can say that (the set) V is a vector space over F (with the addition and scalar being understood).

* Note that we can't decide whether a set V (alone) is a vector space. Vector space is always with respect to a field of numbers F.

Ex: On the set of only one element $V = \{a\}$ and the field $F = \mathbb{C}$, define addition $a + a = a$

and scaling $ca = a \quad \forall c \in \mathbb{C}$.

Then V is a vector space over F. The zero vector is a.

Ex: The set $\mathbb{R}$ is a vector space over $\mathbb{Q}$, but not a vector space over $\mathbb{C}$. Indeed, property (S0) is not satisfied:

$$\underset{\mathbb{C}}{i}\,\underset{\mathbb{R}}{3} \notin \mathbb{R}$$

- How do we check if a set V is a vector space over F?

  Check all the properties (A), (S), (I).

  Note that each property must be true for <u>all</u> x, y, c, d.

  Because of so many requirements, it is generally hard for a set to become a vector space.

- How to show that V is <u>not</u> a vector space over F?

  Give a counter example showing that one of the properties (A), (S), (I) is violated.

<u>Ex</u>: The set of $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{C},\ a + 2b = 0 \right\}$

is a vector space over $\mathbb{C}$.

Indeed, we can check that all vector space axioms are true:

* Check (A0):

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ be in V.

Then $a + 2b = 0$ and $e + 2f = 0$.

The usual addition of matrices gives

$$A + B = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

In order for $A+B$ to belong to V, we check if

$$(a+e) + 2(b+f) = 0$$

$$\text{LHS} = \underbrace{(a+2b)}_{0} + \underbrace{(e+2f)}_{0} = 0$$

* Properties (A1), (A2) hold because of the addition of matrices is commutative and associative.

* Check (A3):

The zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ belongs to V.

This is the zero element of V.

* Check (A4):

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an element of V.

Then $a+2b=0$. The matrix $-A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

also belongs to V because $-a + 2(-b) = -(a+2b) = 0$.

$-A$ is the additive inverse of A because $(-A) + A = 0$.

* Check (S0):

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an element of V.

Let $\alpha \in \mathbb{C}$. We want to check if $\alpha A \in V$.

$$\alpha A = \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$$

We see that $\alpha a + 2(\alpha b) = \alpha \underbrace{(a+2b)}_{0} = 0$

Therefore, $\alpha A \in V$.

* Properties (S1), (S2), (I1), (I2) hold due to the known properties of matrix addition and scaling.

Conclusion: V is a vector space over $\mathbb{C}$.

* Practice on writing proof: let V be a vector space on F which is either $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

1) Show that V has only one zero element.
2) Let $v \in V$. Suppose $v = -v$. Show that $v = 0$.
3) Let $c \in F$, $c \neq 0$ and $v \in V$. Suppose $cv = 0$. Show that $v = 0$.

Proof of 1) :

Let u and w be zero elements of V. Then

By (A3) : $u + v = v \quad \forall v \in V \qquad (1)$

By (A3) : $w + v = v \quad \forall v \in V \qquad (2)$

In (1), choose $v = w$: $u + w = w \qquad (3)$

In (2), choose $v = u$: $w + u = u \qquad (4)$

By (A1), $u + w = w + u$. Therefore, by (3) and (4) we conclude $w = u$.