Lecture 27

Wednesday, December 4, 2019

Let V be a vector space over $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. How do we define distance on V?

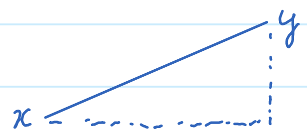
The notion of distance will induce the notion of "closeness" (or convergence), which is the central notion of calculus. For example, with the concept of notion we can define what it means for a sequence of element x_n in V to converge to some x in V.

If $V = \mathbb{R}^n$, the distance between two points

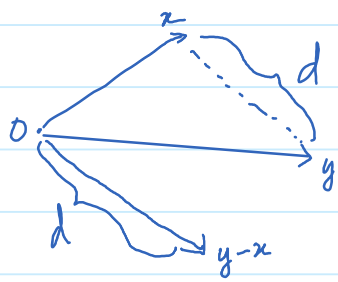
$$x = (x_1, x_2, \ldots, x_n),$$
$$y = (y_1, y_2, \ldots, y_n)$$

can be defined as $\text{dist}(x,y) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$.

This definition is inspired by the Pythagorean theorem.



Intuitively, to define the distance between two general points x and y in V, it suffices to only define the distance between a general point z to the origin 0. Then the distance between x and y can be defined as the distance between $z = y - x$ and 0.



The distance between a vector x to the origin is called the length, or magnitude, or norm of x, denoted by $\|x\|$. We focus on defining the norm of a vector in V.

The norm of x in case $V = \mathbb{R}^n$ can be defined as

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

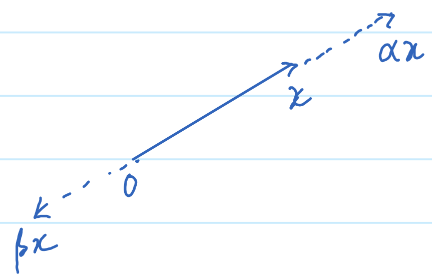
Such definition (by giving an explicit formula) has a drawback: it doesn't tell us what properties the norm satisfies. For example, it is natural to expect that

(1) The norm of a vector is nonnegative:

$$\|x\| \geq 0 \quad \forall x \in V$$

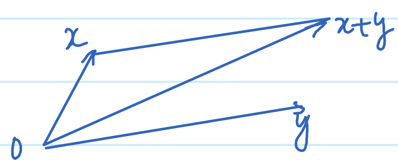
If $\underbrace{\|x\| = 0}_{\text{zero distance from the origin}}$ then $x = 0$.

(2) The norm of a scaling of a vector is a scaling of the norm of the vector by essentially the same factor.



$$\|\lambda x\| = |\lambda| \, \|x\| \quad \forall x \in V, \lambda \in F$$

(3)



$$\|x\| + \|y\| \geq \|x+y\|$$
(triangle inequality)

For this reason, we will define norm by axioms. Recall that we also used an axiomatic approach to generalize $\mathbb{R}^n$ to an abstract vector space.

<u>Definition</u>: Let V be a vector space over a field of numbers $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. A map $\phi : V \to [0, \infty)$ is called a <u>norm</u> on V if it satisfies the following properties:

- Positivity: $\begin{cases} \phi(x) \geq 0 & \forall x \in V, \\ \text{If } \phi(x) = 0 \text{ then } x = 0 \end{cases}$
- Homogeneity:

$$\phi(\lambda x) = |\lambda| \, \phi(x) \quad \forall x \in V, \lambda \in F$$

Note that if $F = \mathbb{C}$ then $|\lambda|$ is the magnitude of complex number λ.

- Triangle inequality:

$$\phi(x+y) \leq \phi(x) + \phi(y) \quad \forall x,y \in V$$

If ϕ is a norm on V, one can write $\|x\|$ instead of $\phi(x)$.

Ex: $V = \mathbb{R}^2$

$$\phi: \mathbb{R}^2 \to [0,\infty), \quad \phi(x_1,x_2) = \sqrt{x_1^2+x_2^2}$$

Show that ϕ is a norm on $\mathbb{R}^2$.

We need to check 3 properties:

(1) Check positivity:

It is clear that $\phi(x) \geq 0$ for all $x \in \mathbb{R}^2$. If $\phi(x) = 0$ then $\sqrt{x_1^2+x_2^2} = 0$. Thus, $x_1 = x_2 = 0$.

(2) Check homogeneity:

Let $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^2$. Write $x = (x_1,x_2)$. Then

$$\begin{aligned}\phi(\lambda x) = \phi(\lambda x_1, \lambda x_2) = \sqrt{(\lambda x_1)^2+(\lambda x_2)^2} &= \sqrt{\lambda^2(x_1^2+x_2^2)} \\ &= |\lambda| \sqrt{x_1^2+x_2^2} \\ &= |\lambda|\, \phi(x).\end{aligned}$$

(3) Check triangle inequality:

We use the following inequality (known as Boniakowsky's inequality or simply "triangle inequality")

$$\sqrt{(a_1+a_2+\dots+a_n)^2+(b_1+b_2+\dots+b_n)^2} \leq \sqrt{a_1^2+b_1^2} + \sqrt{a_2^2+b_2^2} + \dots + \sqrt{a_n^2+b_n^2}$$

for all $a_1, a_2, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$.

Ex:

One can check that $\phi_1 : \mathbb{R}^2 \to \mathbb{R}$, $\phi_1(x_1,x_2) = |x_1| + 2|x_2|$ is also a norm on $\mathbb{R}^2$. But the map $\phi_2 : \mathbb{R}^2 \to \mathbb{R}$,

$$\phi_2(x_1,x_2) = |x_1+2x_2|$$

is not a norm because it doesn't satisfy the positivity property. In fact, $\phi(2,-1) = 0$ but $(2,-1) \neq (0,0)$.