Lecture 26

Monday, December 2, 2016

To check if a linear map $f: V \to V$ is diagonalizable, one can follow the below steps:

1) Find all distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$.
2) For each k, find a basis of the eigenspace $E(\lambda_k)$, called B_k.
3) If $\dim V = \dim E(\lambda_1) + \dots + \dim E(\lambda_m)$ then we get $V = E(\lambda_1) \oplus \dots \oplus E(\lambda_m)$.

Therefore, f is diagonalizable. To get a basis that diagonalizes f, we concatenate the bases $B_1, B_2, \dots, B_m$:

$$B = B_1 \sqcup B_2 \sqcup \dots \sqcup B_m.$$

Then

$$\underbrace{[f]_{B,B}}_{\text{denoted by } [f]_B} = \begin{bmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_1 & & & & & \\ & & & \lambda_2 & & & & \\ & & & & \ddots & & & \\ & & & & & \lambda_2 & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_m \\ & & & & & & & & \ddots \\ & & & & & & & & & \lambda_m \end{bmatrix}$$

To get a decomposition of V into 1-dim invariant subspaces, we further decompose $E(\lambda_k)$ if its dimension is bigger than 1.

For example, if $V = E(\lambda_1) \oplus E(\lambda_2)$ and

$E(\lambda_1)$ has basis $B_1 = \{v_1\}$,
$E(\lambda_2)$ has basis $B_2 = \{v_2, v_3\}$,
then
$$V = \underbrace{E(\lambda_1)}_{1D} \oplus \underbrace{E(\lambda_2)}_{2D} = \underbrace{\underbrace{E(\lambda_1)}_{1D} \oplus \underbrace{Fv_2}_{1D} \oplus \underbrace{Fv_3}_{1D}}_{\text{each is invariant under } f.}$$

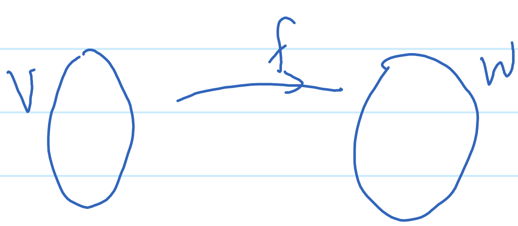
Put $B = B_1 \sqcup B_2 = \{v_1, v_2, v_3\}$. Then
$$[f]_B = \begin{bmatrix} | & | & | \\ [f(v_1)]_B & [f(v_2)]_B & [f(v_3)]_B \\ | & | & | \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}.$$

4) If $\dim V > \dim E(\lambda_1) + \dots + \dim E(\lambda_m)$ then f is not diagonalizable.

Review:

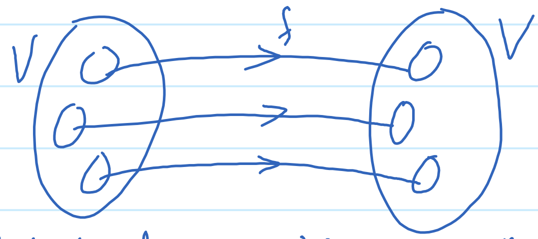
So far we have learned vector spaces, linear maps and some interactions between them.

- vector space has two main ingredients $<$ addition, scaling
- linear maps are a special type of function from a vector space to a vector space that respect the addition and scaling operations.



Linear maps are helpful to study vector spaces. For example, by using the isomorphism that takes a vector v in V to its coordinate $[v]_B \in F^n$, one can translate a problem on V to a problem on F^n.

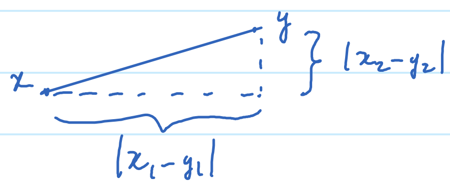
Spectral theory studies an interaction between a vector space and a linear map.



We wanted to decompose V into small pieces, each is one dimensional and is invariant under f.

For any two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, one can define the distance between x and y as

$$\operatorname{dist}(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$



How can we define "distance" in a general vector space?
One way is to define distance through coordinates.
For example, $V = P_2(\mathbb{R})$.

$$u = x^2+1$$

$$v = x+2$$

Let $B = \{x^2, x, 1\}$. Then the coordinate vectors of u and v in basis B are

$$[u]_B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad [v]_B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

One can define the distance between u and v in P_2 as the distance between $[u]_B$ and $[v]_B$ in $\mathbb{R}^3$:

$$\operatorname{dist}(u,v) := \operatorname{dist}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right) = \sqrt{(1-0)^2+(0-1)^2+(1-2)^2} = \sqrt{3}$$

Next time, we will define distance using axioms (like how we defined vector spaces using axioms).