# Lecture 25

Wednesday, November 27, 2019

Let $f: V \to V$ be a linear map. How to check if $f$ is diagonalizable?

* Suppose that $f$ is diagonalizable. Then $V$ can be decomposed into 1-dim invariant subspaces under $f$:

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

Each $V_k$ is the span of $\{v_k\}$ for some $v_k \neq 0$.
$V_k$ corresponds to an eigenvalue $\lambda_k$. $f(v_k) = \lambda_k v_k$. If we put

$$E(\lambda_k) = \{v \in V : f(v) = \lambda_k v\}$$

then $V_k$ is contained in $E(\lambda_k)$. Note that $E(\lambda_k)$ is called the eigenspace corresponding to eigenvalue $\lambda_k$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be distinct eigenvalues of $f$. We have reasoned that each of $V_1, V_2, \ldots, V_n$ is contained in one of $E(\lambda_1), E(\lambda_2), \ldots, E(\lambda_m)$. Thus,

$$\underbrace{V_1 + V_2 + \cdots + V_n}_{=V} \subset E(\lambda_1) + \cdots + E(\lambda_m)$$

Because the RHS is a subspace of $V$, we get

$$V = E(\lambda_1) + \cdots + E(\lambda_m).$$

Note that the RHS is also a direct product (without any assumption on $f$ except that $f$ is linear). We proved a special case of this in Lecture 20 (11/15/2019).

In conclusion, if $f$ is diagonalizable then $V = E(\lambda_1) \oplus \cdots \oplus E(\lambda_m)$.

Now suppose that $V = E(\lambda_1) \oplus E(\lambda_2) \oplus \cdots \oplus E(\lambda_m)$. We show that $f$ is diagonalizable.

Let $B_k = \{v_1^{(k)}, v_2^{(k)}, \ldots, v_r^{(k)}\}$ be a basis of $E(\lambda_k)$. Then

$$E(\lambda_k) = Fv_1 \oplus Fv_2 \oplus \cdots \oplus Fv_r$$

each is 1-dim and invariant under $f$.

Then

$$V = E(\lambda_1) \oplus E(\lambda_2) \oplus \cdots \oplus E(\lambda_m)$$

$$= \underbrace{\left(Fv_1^{(1)} \oplus \cdots \oplus Fv_{r_1}^{(1)}\right)}_{E(\lambda_1)} \oplus \underbrace{\left(Fv_1^{(2)} \oplus \cdots \oplus Fv_{r_2}^{(2)}\right)}_{E(\lambda_2)} \oplus \cdots \oplus \underbrace{\left(Fv_1^{(m)} \oplus \cdots \oplus Fv_{r_m}^{(m)}\right)}_{E(\lambda_m)}$$

This shows that $V$ is decomposed into 1-dim invariant subspaces.

In conclusion, we have showed:

**Theorem:**

Let $f: V \to V$ be a linear map, where $V$ is a vector space over $F$. Let $\lambda_1, \ldots, \lambda_m$ be all distinct eigenvalues of $f$. Then $f$ is diagonalizable if and only if

$$V = E(\lambda_1) \oplus E(\lambda_2) \oplus \cdots \oplus E(\lambda_m).$$

In problem solving, we only need to check if

$$\dim V = \dim E(\lambda_1) + \cdots + \dim E(\lambda_m).$$

Procedure to check if $f: V \to V$ is diagonalizable:

1) Find all the distinct eigenvalues of $f$, called $\lambda_1, \lambda_2, \ldots, \lambda_m$
2) Find a basis $B_k$ for eigenspace $E(\lambda_k)$.

3) Check if $\dim V = \dim E(\lambda_1) + \dots + \dim E(\lambda_m)$

If they are not equal, conclude that $f$ is <u>not</u> diagonalizable.

If they are equal then $F$ is diagonalizable. The basis

$$B = B_1 \sqcup B_2 \sqcup \dots \sqcup B_m$$

diagonalizes $f$ because $[f]_B$ is a diagonal matrix.

$$[f]_B = \begin{bmatrix} | & | & & | & \\ [f(v_1^{(1)})]_B & [f(v_2^{(1)})]_B & \cdots & [f(v_{r_1}^{(1)})]_B & \cdots \\ | & | & & | & \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & \alpha_1 & & & & \\ & & & \alpha_2 & & & \\ & & & & \ddots & & \\ & & & & & \alpha_2 & \\ & 0 & & & & & \ddots \\ & & & & & & \alpha_m \\ & & & & & & & \ddots \\ & & & & & & & & \alpha_m \end{bmatrix}$$