Lecture 24

Monday, November 25, 2019

Let $f: V \to V$ be a linear map. Here V is a vector space over F.
How do we find the eigenvectors and eigenvalues of f?

We solve for $v \in V$, $v \neq 0$ and $\lambda \in F$ from the equation
$$f(v) = \lambda v$$

Ex:

$F: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $F(u)(x) = u(x+1)$

Find all the eigenvectors and eigenvalues of F.

* Method 1: (direct method)

We solve for $u \in P_3$, $u \neq 0$ and $\lambda \in \mathbb{R}$ from the equation
$$F(u) = \lambda u \qquad (*)$$

Write $u = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$. We have
$$LHS(*) = u(x+1) = a(x+1)^2 + b(x+1) + c$$
$$= ax^2 + (2a+b)x + a+b+c.$$
$$RHS(*) = \lambda(ax^2+bx+c) = \lambda a x^2 + \lambda b x + \lambda c.$$

For LHS to be equal to RHS, we need
$$\begin{cases} a = \lambda a, \\ 2a+b = \lambda b, \\ a+b+c = \lambda c. \end{cases}$$

This system is equivalent to
$$\begin{cases} (1-\lambda) a = 0 \\ (1-\lambda) b = -2a \\ (1-\lambda) c = -a - b \end{cases}$$

- If $\lambda \neq 1$ then we get
$$\begin{cases} a = 0, \\ b = -2a/(1-\lambda) = 0, \\ c = (-a-b)/(1-\lambda) = 0. \end{cases}$$

In this case, $u = ax^2 + bx + c = 0$. Thus, any $\lambda \neq 1$ is <u>not</u> an eigenvalue of F.

- If $\lambda = 1$ then

$$\begin{cases} a \in \mathbb{R} \\ 0 = -2a \\ 0 = -a-b \end{cases}$$

This gives $a = b = 0$ while $c \in \mathbb{R}$ is arbitrary.
Thus, $u = c$

We conclude that $\lambda = 1$ is the only eigenvalue of F
$u_0 = 1$ is an eigenvector corresponding to $\lambda = 1$. Any other eigenvector of F is a nonzero multiple of u_0.

* Method 2 : (use coordinates to work with matrices)
We try to convert the problem of finding eigenvectors/eigenvalues of F to a problem of finding eigenvectors/eigenvalues of a matrix, which is the type of problems we were more familiar with in Linear Alg. I.

$V = P_2(\mathbb{R})$ is a vector space of dimension 3.
V has a basis $\mathcal{B} = \{\underbrace{x^2}_{e_1}, \underbrace{x}_{e_2}, \underbrace{1}_{e_3}\}$.

Any vector $u = ax^2 + bx + c \in P_2$ has coordinate vector in basis $\mathcal{B}$ given as:

$$[u]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Linear map F is represented by matrix

$$[F]_{\mathcal{B}} = \begin{bmatrix} | & | & | \\ [F(e_1)]_{\mathcal{B}} & [F(e_2)]_{\mathcal{B}} & [F(e_3)]_{\mathcal{B}} \\ | & | & | \end{bmatrix}$$

We have

$F(e_1) = F(x^2) = (x+1)^2 = x^2 + 2x + 1$

Thus, $$[F(e_1)]_B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$F(e_2) = F(x) = x + 1.$

Thus, $$[F(e_2)]_B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$F(e_3) = F(1) = 1.$

Hence, $$[F(e_3)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $$[F]_B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Next, we find the eigenvectors and eigenvalues of this matrix (called A).

$$A - \lambda I_3 = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{bmatrix}$$

Because this is a lower triangular matrix,

$$\det(A - \lambda I_3) = \underbrace{(1-\lambda)(1-\lambda)(1-\lambda)}_{\text{product of entries on the diagonal}} = (1-\lambda)^3.$$

Thus, A has only one eigenvector $\lambda = 1$.

Let us find all eigenvectors associated with $\lambda = 1$.

$$A - I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The equation $(A - I_3)v = 0$ has infinitely many solutions

$v = (0, 0, c)$ with $c \in \mathbb{R}$.

Let us convert the problem back to P_2:

$v = (0, 0, c) \in \mathbb{R}^3$ corresponds to $u = c \cdot 1 = c \in P_2$.

Therefore,

$$\begin{cases} \lambda = 1 \text{ is only eigenvalue of } F \\ u = c \text{ (constant function) for any } c \neq 0 \text{ is an eigenvector of } F. \end{cases}$$