Lecture 23

November 22, 2019

Let $f: V \to V$ be a linear map.
We want to know whether V can be decomposed as

$$V = V_1 \oplus V_2 \oplus \dots \oplus V_m$$

where each V_k is invariant under f and of small dimensions.

* If each $V_1, V_2, \dots, V_m$ is one-dimensional then f is said to be <u>diagonalizable</u>. Note that in this case, $\dim V = \dim V_1 + \dots + \dim V_m = 1+1+\dots+1 = m$.
Let $\{v_k\}$ be a basis of V_k. Since V_k is invariant under f, $f(v_k) \in V_k$. Thus,

$$f(v_k) = \alpha_k v_k$$

for some $\alpha_k \in F$.

v_k is called an <u>eigenvector</u> and α_k is called an <u>eigenvalue</u>.

Note that by this definition, an eigenvector is always nonzero. An eigenvalue can be zero. Also, any nonzero multiple of an eigenvector is an eigenvector.

Because of the direct sum $V = V_1 \oplus V_2 \oplus \dots \oplus V_m$, we see that the concatenation $B = B_1 \sqcup B_2 \sqcup \dots \sqcup B_m = \{v_1, v_2, \dots, v_m\}$ is a basis of V. It is a basis that <u>diagonalizes</u> f. We use the term "diagonalize" because the matrix representation of f with respect to basis B is diagonal.

$$f: \underbrace{V}_{\text{basis } B} \longrightarrow \underbrace{V}_{\text{basis } B}$$

$$[f]_{B,B} = \begin{bmatrix} | & | & & | \\ [f(v_1)]_B & [f(v_2)]_B & \cdots & [f(v_m)]_B \\ | & | & & | \end{bmatrix}$$

Because $f(v_1) = \alpha_1 v_1 = \alpha_1 v_1 + 0 v_2 + \dots + 0 v_m$, we have

$$[f(v_1)]_B = \begin{bmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Because $f(v_2) = \alpha_2 v_2 = 0 v_1 + \alpha_2 v_2 + 0 v_3 + \dots + 0 v_m$, we have

$$[f(v_2)]_B = \begin{bmatrix} 0 \\ \alpha_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Do similarly to each column. We get

$$[f]_{B,B} = \begin{bmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_m \end{bmatrix}$$ which is a diagonal matrix.

For simplicity, we will often write $[f]_B$ instead of $[f]_{B,B}$.

<u>Ex</u>: Let $F: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $F(u) = u(2x)$.
Find all the eigenvectors and eigenvalues of F.

To find the eigenvectors and eigenvalues of F, we solve for $u \neq 0$ and $\lambda \in \mathbb{R}$ such that
$$F(u) = \lambda u. \quad (*)$$

Let us write $u = ax^2 + bx + c$. Then
$$LHS(*) = F(u) = u(2x) = a(2x)^2 + b(2x) + c = 4ax^2 + 2bx + c.$$
$$RHS(*) = \lambda u = \lambda(ax^2 + bx + c) = \lambda a x^2 + \lambda b x + \lambda c$$

For $LHS(*)$ to be equal to $RHS(*)$, we must have
$$\begin{cases} 4a = \lambda a, \\ 2b = \lambda b, \\ c = \lambda c. \end{cases}$$

This system is equivalent to
$$\begin{cases} (4-\lambda)a = 0 \\ (2-\lambda)b = 0 \\ (1-\lambda)c = 0. \end{cases}$$

If $\lambda \neq 1, 2, 4$ then the system gives $a = b = c = 0$. This corresponds to $u = 0$.
Note that $u = 0$ is not an eigenvector.

If $\lambda = 4$ then $b = c = 0$ while a is arbitrary. Then $u = ax^2$.
Thus, $\lambda = 4$ is an eigenvalue and $u_1 = x^2$ is an eigenvector associated with $\lambda_1 = 4$.
Any other eigenvector corresponding to λ_1 is a nonzero scalar multiple of λ_1.

If $\lambda = 2$ then $a = c = 0$ while b is arbitrary. Then $u = bx$.
We conclude that $\lambda_2 = 2$ is an eigenvalue corresponding to eigenvector $u_2 = x$.
Any other eigenvector corresponding to λ_2 is a nonzero scalar multiple of λ_2.

If $\lambda = 1$ then $a = b = 0$ while c is arbitrary. Then $u = c$.
We conclude that $\lambda_3 = 1$ is an eigenvalue corresponding to eigenvector $u_1 = 1$.

If $\lambda = 1$ then $a = b = 0$ while c is arbitrary. Then $u = c$.
We conclude that $\lambda_3 = 1$ is an eigenvalue corresponding to eigenvector $u_1 = 1$.
Any other eigenvector corresponding to λ_3 is a nonzero scalar multiple of λ_3.