Lecture 22 (11/20/2019)

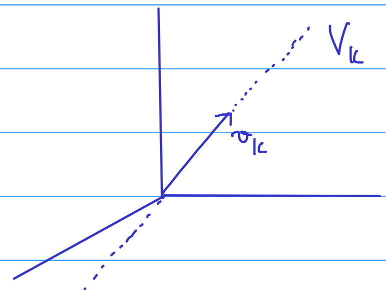
* Spectral theory: let V be a vector space over a field $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Suppose $f : V \to V$ is a linear map. We ask whether V can be decomposed as $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ where each V_k is an invariant subspace under f.

Of course one can write $V = V \oplus \{0\} \oplus \cdots \oplus \{0\}$. Each component is invariant under f. However, this decomposition is trivial in the sense that it doesn't exploit any properties of f. In general, we prefer $V_1, V_2, \ldots, V_n$ to be of small dimensions.

* Suppose each V_k is one-dimensional with basis $B_k = \{v_k\}$. Because $f(v_k) \in V_k$, we can write $f(v_k) = \alpha_k v_k$ for some $\alpha_k \in F$.

Then for any $w \in V_k$, we also have $f(w) = \alpha_k w$. Indeed, each $w \in V_k$ can be expressed as $w = c v_k$ for some $c \in F$. By the scalar multiplicative property of f,

$$f(w) = f(c v_k) = c f(v_k) = c \alpha_k v_k = \alpha_k w.$$



Therefore, f transforms each vector in V_k by scaling by factor α_k. In practical, f preserves the direction of every vector in V_k.

Another observation: (still under the assumption that each V_k is one-dimensional)

Each $v \in V = V_1 + \cdots + V_n$ can be written as

$$v = c_1 v_1 + \cdots + c_n v_n$$

Applying f to both sides, we get

$$f(v) = f(c_1 v_1 + \cdots + c_n v_n)$$

$$= c_1 f(v_1) + \cdots + c_n f(v_n)$$
$$= c_1 \alpha_1 v_1 + \cdots + c_n \alpha_n v_n.$$

Let us use the notation $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$.

Then $f^2(v) = f(f(v)) = f(c_1\alpha_1 v_1 + c_2\alpha_2 v_2 + \cdots + c_n \alpha_n v_n)$
$$= c_1 \alpha_1^2 v_1 + c_2 \alpha_2^2 v_2 + \cdots + c_n \alpha_n^2 v_n.$$

Similarly, for any positive interger k,

$$f^k(v) = c_1 \alpha_1^k v_1 + c_2 \alpha_2^k v_2 + \cdots + c_n \alpha_n^k v_n. \qquad (*)$$

We see that the linear map f^k is expressed very simply in terms of the basis vectors $\{v_1, \ldots, v_n\}$ of V.

We know that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

for any $x \in \mathbb{R}$. Define formally

$$e^f = Id + \frac{f}{1!} + \frac{f^2}{2!} + \frac{f^3}{3!} + \cdots$$

In other words, we define "e^f" as a map that does the following: for each $v \in V$,

$$(e^f)(v) := v + \frac{f(v)}{1!} + \frac{f^2(v)}{2!} + \frac{f^3(v)}{3!} + \cdots$$

Let us apply $(*)$ to the right hand side. After formally reorganizing the terms, we get

$$(e^f)(v) = c_1\left(1 + \frac{\alpha_1}{1!} + \frac{\alpha_1^2}{2!} + \cdots\right) v_1 + c_2\left(1 + \frac{\alpha_2}{1!} + \frac{\alpha_2^2}{2!} + \cdots\right) v_2$$

$$\cdots + c_n\left(1 + \frac{\alpha_n}{1!} + \frac{\alpha_n^2}{2!} + \cdots\right) v_n$$

$$= c_1 e^{\alpha_1} v_1 + c_2 e^{\alpha_2} v_2 + \cdots + c_n e^{\alpha_n} v_n$$

Note that we have defined a linear map e^f from the linear map f thanks to the decomposition

$$V = V_1 \oplus \dots \oplus V_n \text{ in which } \begin{cases} \text{each } V_k \text{ is 1-dim,} \\ V_k \text{ is invariant under } f. \end{cases}$$

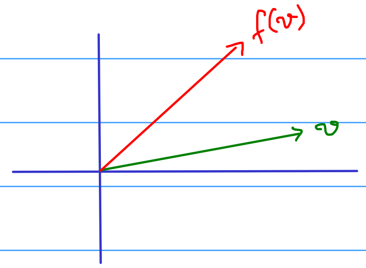
Spectral theory is mainly about the topic: given a vector space V over F and a linear map $f: V \to V$. can we find a decomposition $V = V_1 \oplus \dots \oplus V_n$ such that each component is invariant under f. We prefer each $V_1, V_2, \dots, V_n$ to be of small dimension.

Spectral theory plays an important role in operator theory (e.g. the above example).

* Theorem:

Let V be a finite dimensional vector space over $F = \mathbb{C}$. Let $f: V \to V$ be a linear map. Then V has at least one 1-dimensional subspace that is invariant under f.

Note that the theorem would not be true if $F = \mathbb{R}$. One can in fact find a counterexample in the case $V = \mathbb{R}^2$.



Let f be a rotation mapping on $\mathbb{R}^2$. For example, a $+90°$ rotation corresponds to

$$f(x,y) = (-y, x).$$

Unless the angle of rotation is a multiple of $180°$, the rotation f doesn't preserve any direction. Therefore, there are no 1-dim subspace of $\mathbb{R}^2$ (i.e. lines) that are invariant under f.

However, the theorem is true when $F = \mathbb{C}$. We will see that this is due to the fact that any polynomial with coefficients in $\mathbb{C}$ has at least one root in $\mathbb{C}$.

* Analysis of proof to the theorem:

Let us consider the case $V = \mathbb{C}^2$.

Suppose W is a 1-dim subspace of $\mathbb{C}^2$ that is invariant under f. Because $\dim W = 1$, W must contain some nonzero vector, called v_0. We have $f(v_0) \in W$ because W is invariant under f. Then

$$f(v_0) = \lambda v_0$$

for some $\lambda \in \mathbb{C}$. One can rewrite the equation as

$$f(v_0) = \lambda \, id \, (v_0)$$

or

$$(\underbrace{f - \lambda \, id}_{g})(v_0) = 0.$$

We see that $v_0 \in null(g)$. Because $null(g)$ contains a nonzero vector v_0, g is not monomorphic. Thus, g is not isomorphic.

Let us fix a basis of $V = \mathbb{C}^2$, for example the standard basis $B = \{e_1, e_2\} = \{(1,0), (0,1)\}$. Then

$$[g]_B = [f - \lambda\, id]_B = [f]_B - \lambda [id]_B$$
$$= [f]_B - \lambda I_2$$

Because g is not isomorphic, its representation matrix $[g]_B$ is not invertible. This is equivalent to $\det [g]_B = 0$.
Then

$$\det([f]_B - \lambda I_2) = 0.$$

Let's write $[f]_B = \begin{bmatrix} | & | \\ f(e_1) & f(e_1) \\ | & | \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

for some $a, b, c, d \in \mathbb{C}$. This is considered to be a known matrix because f is a given function. Then

$$\det([f]_B - \lambda I_2) = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$
$$= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$$
$$= \text{quadratic polynomial of } \lambda.$$

This polynomial always has a complex root $\lambda = \lambda_0$. Note that this is the case only when we work on $\mathbb{C}$, not $\mathbb{R}$. We conclude the analysis of the proof.

*Rigorous proof of the theorem:
The strategy is to go backward from the above analysis. We start from the equation $\det([f]_B - \lambda I_2) = 0$. This is a polynomial of λ, thus having a complex root $\lambda_0 \in \mathbb{C}$. Put $g = f - \lambda_0\, id$. Then

$$\det [g]_B = \det [f - \lambda_0\, id]_B$$
$$= \det([f]_B - \lambda_0 I_2)$$
$$= 0.$$

Hence, $[g]_B$ is not invertible. This implies g is not isomorphic. Since g goes from V to V, isomorphism is the same is monomorphism. Then g is not monomorphic. That means $\text{null}(g) \neq \{0\}$. Thus, $\text{null}(g)$ must contain some nonzero vector v_0. We have

$$\underbrace{g(v_0)}_{f(v_0) - \lambda_0 v_0} = 0$$

Then $f(v_0) = \lambda_0 v_0$. Therefore, the space $W = \text{span}\{v_0\}$ is invariant under f.